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ENGINEERING KINEMATICS

BY

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PREFACE

The increasing importance of the subject of dynamics to engineers demands a rigorous exploration of the division of kinematics. In the pedagogical time-table this division most frequently appears in the second year, and in the role of a course in Mechanism.

The aims and objectives of the mechanism courses concern themselves primarily with preparation for Machine Design and incidentally borrow, as needed, from the formal courses in Applied Mechanics. A series of elements of mechanism is investigated, and their kinematic significance is too frequently subordinated to descriptive material as the individual packages of elements of mechanism are collected.

A more solid base can be built—there are ever-present fundamentals of displacement, velocity, and acceleration, which furnish the common background of all mechanisms.

These concepts of kinematics are the actual and broader elements of mechanism. Cams, gears, and the other applications are but their applied form.

This text intends to pursue an objective of training in fundamentals and has therefore been constructed upon the bases of displacement, velocity, and acceleration. Mechanisms appear as illustrations, not as divisions of the subject of Engineering Kinematics. They can thus be thoroughly and effectively explored, and their appearance, at first as illustrations of displacement, again in applications of velocity, and later as the field of study of accelerations gives coordination to the development of strength in the study of kinematics.

A course in Engineering Kinematics should serve in a liaison capacity between principles expounded in Applied Mechanics, and practices explored in Machine Design. It can enhance further training in both fields.

The courses in Applied Mechanics cannot, in the portion devoted to kinematics, thoroughly explore the applications of principle to specific mechanisms; the courses in Machine Design should not advance to a consideration of the stress distribution, choice of material, methods of lubrication of machine parts before a real foundation of the kinematic significance of those parts has been laid.

The author does not claim any originality in the choice of material from the field usually covered in the courses in Mechanisms.

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The departure from tradition has been pedagogical—none of the elements of mechanism are explored before a solid foundation of the general

principles of kinematics has been built. The objective first sought is a thorough mastery of displacement, velocity, and acceleration. If these properties are sincerely and thoroughly investigated, a background from which all elements of mechanism stem has been established. This objective precludes the usual division into chapters devoted to "Cams," "Gears," et cetera, since such elements in general illustrate applications of more than one kinematic property: they can be used most effectively and with more telling effect as illustrations of the several properties rather than as entities. They must therefore be offered for consideration in more than one role. The traditional division of the subject into chapters based upon the mechanism elements fails to foster the development of strength in the common kinematic background which these elements share.

Such a division, logical and penetrating, is desirable, even though the chapter devoted to velocity may seem at first to be excessively long. The property of velocity furnishes the major portion of the axiomatic principle upon which the theory of mechanisms rests, and its discussion must, therefore, occupy a major portion of the allotted space.

Attention is called to the manner in which a strong undercurrent of basic principle should be established, and constantly forced to bulwark the development. The vector discussion crystallizes in the very potent orthogonal component—a basic weapon. A solid concept of absolute and relative motion is next added. The entire discussion of velocity, of acceleration, and hence of the mechanisms which serve as outlets for these properties can then develop with the rigid logic of growth from a single base. This method is the antithesis of chapter-shots fired now at cams, then at gears,—a system which must always lack strength in training in kinematic principle, since the cam or the gear becomes the direct object, and not the eventual expression of principle.

Such pedagogy may be appropriate for the "practical" man of the trade school; it cannot play its part in the more thorough development of the engineering student, who should in all of his training be seeking first for mastery of fundamental principle, and later for the evolution into practical or applied form.

Notation systems have been employed only sparingly. While some use of subscripts and exponents is justifiable because absolute avoidance of these abbreviations would result in extremely long and awkward description, the use of involved systems of symbols disturbs the student unnecessarily.

Too frequently college students evade the prefaces of their textbooks. Prefatory remarks are intended to point the way, and to serve as an invitation from author to reader to share in a mutual effort. In that spirit this preface is continued, as Chapter I of the text.

The author wishes to express his appreciation of the assistance received from his colleagues at the Massachusetts Institute of Technology, and particularly to Mr. Deane Lent of the Institute, who furnished many valuable suggestions and contributed many of the illustrations.

A. S.

Cambridge, Mass.
January, 1941

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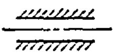
ABBREVIATIONS AND SYMBOLS

a	linear acceleration
deg.	degrees
ft.	feet
f.p.m.	feet per minute
f.p.s.	feet per second
in.	inches
m.p.h.	miles per hour
min.	minutes
rev.	revolutions
r.p.m.	revolutions per minute
r.p.s.	revolutions per second
s	linear displacement
sec.	seconds
t	time
t.v.	train value
v	linear velocity
α	angular acceleration
θ	angular displacement
ω	angular velocity
A/B	A relative to B
$+\!\!\!>$	plus, vectorially
$-\!\!\!>$	minus, vectorially
	pin joint
	pin joint, fixed axis
	fixed guides

ENGINEERING KINEMATICS

I

INTRODUCTION

The study of applied mechanics plays a dual role in engineering education. It has direct tool value as it furnishes a basis for the design of machines or of structures. It also affords opportunity for training in the development of mental power—in "straight thinking." While philosophical speculation may furnish a background of theorem or axiom, the application of principles to concrete problems demands planned, orderly, effective reasoning.

The student of elementary statics, for example, makes tangible progress only when he learns that each attack upon unfamiliar frames or trusses becomes successful when the same basic analytical reasoning is applied—the logical selection of a free body which must be isolated from its neighbors and set up as a center of interest; the assigning to that free body of the system of external forces, both known and unknown, which act upon it; and finally, the application of the proper conditions of equilibrium to a solution for the unknowns.

The insistence upon *order* in thinking is a characteristic to which each subject in the education of an engineer should contribute.

The history of the development of courses in kinematics has not paralleled the logical development of the teaching of statics. Such development, in general, emphasized the study of a series of mechanisms, and only occasionally has an effort been made to develop first a fundamental equipment of the basic kinematic principles upon which all design of mechanisms rests.

At the same time there has evolved an increasing necessity for training in the kinematics of motions involving greater attention to the property of acceleration than was formerly necessary. The engineering student must face such problems with confidence rather than with the timidity that usually characterizes encounters with the accelerations of dynamics problems. Such confidence can rest only upon mastery of fundamentals.

In the present text, the effort is directed at building the solid base; applications to mechanisms are explored only after bases have been established.

There are inherent in this subject the same possibilities of logical growth of analytical reasoning which enhance other branches of training in applied mechanics.

For example, the investigations made in kinematics are very largely based upon the use of vectors. To establish strength and confidence in the use of these quantities, their fundamental properties have been established in the form of two basic theorems. Similarly, absolute and relative motion concepts have been crystallized into the definite form of a third basic theorem.

Some discussions of fundamental concepts are elaborated to an extent which may seem to be unnecessarily cautious and painstaking, but this is intended to be a textbook, not a learned treatise, and class-room experience has convinced the author that there is no encouragement to mental activity on the part of the student when instruction evades its responsibility of guidance by falling back, in cases apparent to the instructor, upon the classical and hackneyed "It is obvious that" . . .

In other instances, the classic style has been avoided; there seems to be no tenable argument for couching all texts in the same subject in identical terms, even though time has hallowed the usage.

The device of reiteration has been used freely. The impact of a new idea is resisted as frequently by the student as by individuals in other walks of life. This resistance becomes weaker and finally capitulates if the new idea is permitted to offer its attack repeatedly.

Education may be evaded almost as readily within academic walls as without; the coming of intellectual maturity is not always coincidental with the chronological age of college students.

In engineering education, however, there are ever present opportunities of encouragement to intellectual awakening, to the joys of scholarship, and to the satisfaction which comes with constructive essays into straight thinking.

In this field of applied kinematics, there can be fostered attitudes of devotion to the development of an effective mind. This is a very different approach to professional success than the search for formulas and type- or case-problems in mechanisms which, learned through discipline and rote, make of any form of education a tragic farce.

The student is urged to use this text as a friendly guide, to accept its statements only after honest questioning, and never to assume that they come as divine revelations which must be accepted with humble submission.

II

THE RIGID BODY OF MECHANICS

1. The Rigid Body. The bodies whose properties of motion are to be explored are assumed to be rigid, that is, non-deforming, under the action of the forces which produce the motion.

The definition of the rigid body of mechanics takes, then, this form: *A rigid body is one whose particles remain constantly at the same distance from each other.*

Material bodies are never so rigid that all of the particles will completely resist being displaced relative to each other. A division of applied mechanics—the study of the Strength of Materials—concerns itself very largely with such relative displacements.

When a body is set in motion, however, the possible relative displacements of particles are so small in comparison with the displacement of the body as a whole that they may be neglected.

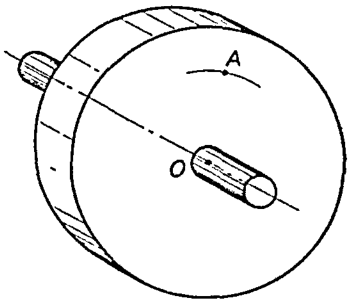


FIG. 1

2. The Rigid Body of Kinematics. The size of the rigid body of kinematics requires a somewhat different concept than the usual description of a solid body of fixed dimensions.

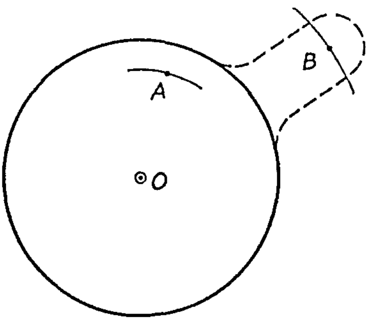


FIG. 2

The wheel shown in Fig. 1 has definite dimensions—diameter and thickness—which establish its physical extent.

In kinematics, however, the wheel is considered but part of a body of indefinite extent.

It will be noted that any particle of the wheel, like A, is forced to turn in a circular path about the axis O, at a speed which is determined by the speed of the wheel.

If, as in Fig. 2, the wheel were to be increased in size so that it contained a particle like B, then B would also be forced to turn in a circular path

about O and its speed would be fixed by the motion given to the wheel. The distance of particle B from A and from O would always remain unchanged, for this is a rigid body.

The physical size of the wheel need not be increased to enable it "kinematically" to contain particle B. If any particle like B is moving so that its distance from A and from O remains unchanged during the motion of the wheel, then this particle belongs, as far as motion is concerned, to the wheel, just as truly as if the wheel were physically expanded.

This "kinematic expansion" of the size of the rigid body is subject to no limits of size, but is restricted in that every particle assigned to the rigid body must have the proper motion properties possessed by particles of the rigid body.

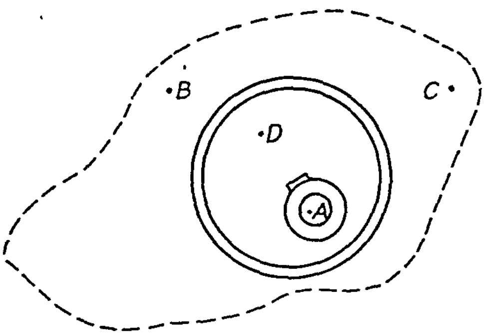


Fig. 3

The eccentric wheel of Fig. 3 is a body which kinematically is unlimited in size and includes not only particles which like A and D are physically present in the material of the eccentric wheel, but any other particles, such as B and C, whose motion is such that they move when the wheel moves, and remain ever at constant distance from A and D and from each other.

This attachment of new particles to the original body which are to be carried along with the body as it moves is a necessary device of analysis—and it has practical import in many studies of motion.

In addition to the kinematic expansion of the rigid body, one convenient contraction of its dimensions should be made. The major portion of the subject of kinematics concerns itself with coplanar motion, that is, motion in which all particles of the body move in the same or in parallel planes. For example, in the wheel of Fig. 4, the particles which lie in any plane perpendicular to the axis move in that plane constantly. The description of motion which is developed by considering the particles of any one such plane is repeated exactly

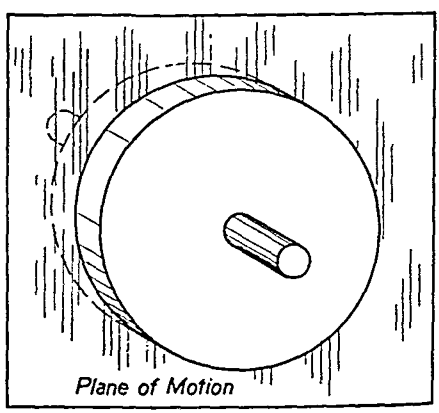


Fig. 4

in all of the planes parallel to it. We may, therefore, simplify our analyses by assuming that the wheel is of one-plane thickness and investigate motion in that one plane, reporting, if necessary, identical conditions in all parallel planes.

The plane to which the thickness is reduced is called the *plane of motion,* and it is customary to select the plane which contains the mass-center of the body, as the plane of motion.

The concept of the "rigid body" of mechanics is as basic in importance as any other equipment we may develop as a tool of analysis.

The reasoning of engineering must never be haphazard, wandering as fancy may dictate, but logical, coherent, and constantly directed. The study of forces acting upon a body, or of the motion of that body, very definitely demands as its starting-point an absolute and accurate identification of the exact body—its nature and limits.

In machines as in structures, the rigid body upon which analytical attacks are focussed is rarely found standing alone, like the specimen blocks or cylinders of an abstract problem—most frequently, the rigid body forms one of a series of connected units. It must, therefore, be carefully isolated from its neighbors. This segregation demands full appreciation of the limits of the body, and of the effect which every neighboring body has upon it.

The abilities to isolate the rigid body, and to assign correctly its properties in the study of kinematics are paralleled in every branch of analysis. The philosophy which is implied of ordered reasoning from a carefully selected, clearly defined base is universally applicable. As a keystone of habit formation, it serves the student of engineering as a most effective agent in the development of the engineering attitude.

III

VECTORS

3. Vector Quantities. The quantities which are encountered in the several branches of mechanics are customarily placed in two divisions: those having magnitude only, called *scalars*, like mass, temperature, and time, and those which are endowed with a property of direction in addition to magnitude. The latter quantities are called *vector quantities*. Force, velocity, and acceleration are typical examples.

The definition of vector quantities suggests the most convenient way of representing them in order to convey clearly a complete and accurate description from one person to another—the graphical means of a drawing. It will be found that this method has other advantages than clarity—in combining vector quantities with other vector quantities or with scalars, graphical operations are efficient, simple, and direct.

4. Graphical Representation of a Vector Quantity. A line, called a *vector*, is the graphical translation of a vector quantity.

Consider as an illustration the following description, in words, of a displacement or change of position: "displacement of five feet along a line

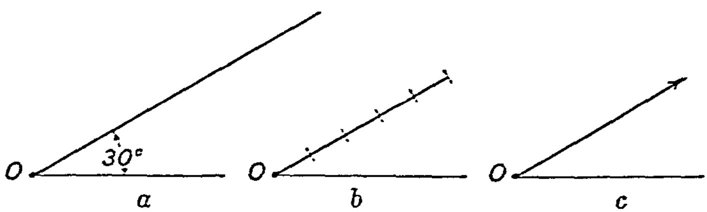


FIG. 5

which makes an angle of 30° with a horizontal axis, and so directed that the change of position is upward and to the right." This description is cumbersome, and if several vectors are to be described, unnecessarily lengthy and awkward.

We turn, therefore, to a drawing and translate the description into a graphical one.

There are three stages in the graphical representation of such a vector quantity as the one described above.

First, as in Fig. 5-a, a line is drawn, having the proper *inclination*, that is,

making an angle of 30° with the horizontal, and originating at the starting point of the displacement, point O, called the *origin* of the vector.

The *magnitude* of the vector quantity, in this case five feet, is next represented by assigning a scale, as 1 division = 1 ft., and letting the length of the drawn line represent the magnitude of the vector quantity, as in Fig. 5-b.

Finally, the *sense* of the vector quantity, which is that characteristic which reveals in which way the change of position is taking place, as "upward to the right" is announced by placing an arrow-head at the end, or *terminus* of the drawn vector, pointed in the direction of travel (Fig. 5-c).

The final drawing is a pictorial description of the vector quantity which is concise, clear, and complete.

It will be noted that the "direction" property of a vector quantity has two aspects: (a) its inclination, or angle with some known reference axis; and (b) its sense, which segregates the intended meaning from the two possibilities of travel along a line of definite inclination.

5. The Resultant. Addition of Vectors. The single vector whose properties have been considered is generally found as one of a group, which is called the system of vectors.

Many systems of vectors are equivalent. For example, there are innumerable combinations of forces which might be applied to a body to give the body one desired direction and magnitude of displacement.

A *resultant* vector quantity is the simplest equivalent system to which a group of vector quantities can be reduced.

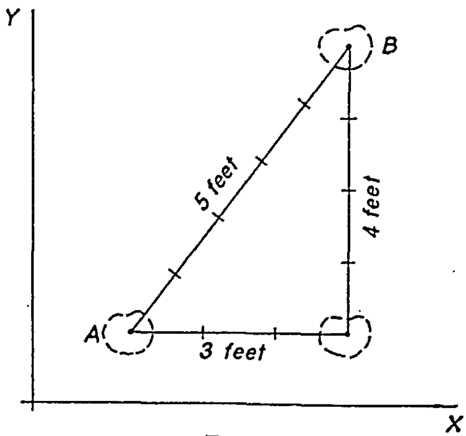


FIG. 6

If the body shown in Fig. 6 is given a displacement of 3 ft. parallel to the X-axis, starting at A, and another displacement of 4 ft. parallel to the Y-axis is then added, the body will finally reach point B, which is 5 ft. from A.

An equivalent final displacement could be accomplished by giving the body one displacement of 5 ft. in the direction AB.

These component parts of the final displacement may be in action during the same time. For example, the piston of an automobile engine may be moving upward relative to the frame of the car at the same time that the car moves forward. Then points on the piston will have resultant or final

displacements in space which are the sum of individual displacements taking place at the same time.

In the case of Fig. 6, a group of two individual displacements of the body, or two vector quantities, have been been added. A displacement which is equivalent in all respects may be accomplished by the most simple system of vector quantities, a single vector. This single displacement is the resultant vector quantity, since it is the simplest system of vectors which will have exactly the same final effect upon the body.

The process of adding the vectors of a group to obtain their resultant,

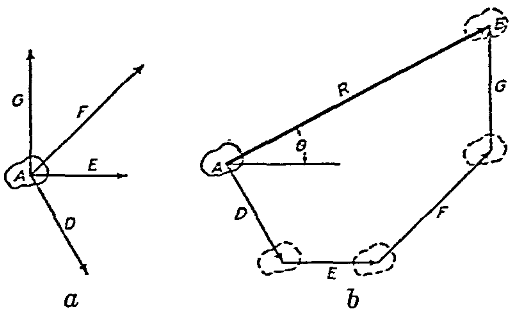


FIG. 7

called *composition*, summarizes the geometrical route of displacement in going from point A to point B in the previous illustration.

The body shown at A in Fig. 7-a is acted upon by several forces at the same time. If each acted alone, it would cause a displacement of the body in the direction and of the magnitude shown, as vectors D, E, F, and G.

The final effect upon the body may be ascertained by summarizing the entire group of displacements, or simply by finding the resultant displacement.

Starting at A, as in Fig. 7-b, the vector representing displacement D is drawn. Next, starting at the terminus of displacement D, the displacement E is added (the terminus of D becomes the origin of E) as are the remaining displacements, which as vectors are all drawn in their announced direction and to the same scale of displacement. The last point reached is point B, and the effect of the group of displacements is to move the body from A to B, which is displacement R, at inclination θ, and of sense shown by the arrow-head. The simplest equivalent system which could be used to accomplish the same purpose is a single vector R which has the same magnitude, inclination, and sense. R is then the resultant of the original system.

The order in which the several vectors are added to each other has no effect upon the final result. Figure 8 shows the addition of the same group

of vectors, adding in the order E, D, G, F, and giving as their sum the vector R, which is identical with the previously obtained resultant.

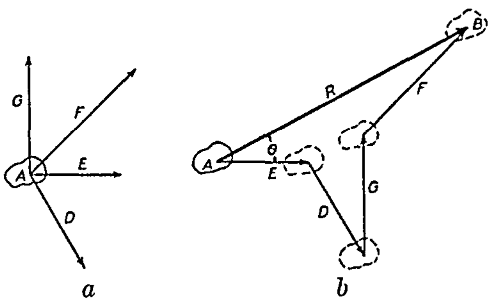


FIG. 8

The independence of the sum—its freedom from the order of addition of the vectors—is called the commutative law of addition.

PROBLEMS

Before attacking the problems of this, the first group of the book, a clear understanding should be established of the methods and techniques which are to be employed.

In most analyses of the motion of mechanisms, graphical solutions have the advantage of being more rapid than numerical or analytical attacks, and if the necessary drafting is precise, there is no disadvantage. The efficiency of the graphical solution becomes particularly evident when a number of vectors are involved in the solution, or when mechanism analyses involve study of the mechanism in several different positions.

On the other hand, if we wish to solve a problem where an analytical equation may readily be written, and numerical values substituted to obtain an exact solution, this road of attack may become the most direct.

Setting an arbitrary rule that either graphical or analytical solutions must be used is as foolhardy as rigorously insisting that a screw-driver must be employed for every operation involving the use of a tool. The screw-driver may be a versatile tool, but it is not the most efficient means of accomplishing the rotation of a nut upon a bolt. The graphical and analytical methods of solution are also tools, and that one should be employed which best fits the case.

Very frequently, one method may be used to amplify or assist the other—this is particularly true in the case of checking—and checking is a sound engineering procedure.

Similarly reports of a problem solution may make use of numerical or graphical descriptions in announcing the answer. For example, in reporting the several different speeds of a piston at various times during its stroke or path of travel, a numerical table may be given. This is far less easily interpreted, however, than a curve or graph in which values of speed may be plotted against time, enabling the reader to note not only speeds at various points, but the nature of changes and the trends in different portions of the stroke.

The determination of a proper plan of attack, and the selection of the most efficient tools and procedures, require the development of judgment, which comes through experience. In the earlier sets of problems the type of attack will be indicated; these will

serve to familiarize the student with the possibilities of the various methods. Later, when familiarity has developed a sense of orientation, the student may select his procedure.

Another practise of the engineer should be employed here. In exploring a problem to crystallize its demands in one's mind, the sketch is an effective agent. One should have a sketch pad at hand, and should size up the problem by working out a rough draft of the plan of attack before entering upon the preparation of the finished drawings or the detailed calculations.

When graphical means are to be used in solving the problem, the results must be obtained from a drawing. Such results are trustworthy only if the drawings which are prepared are executed with a high degree of accuracy.

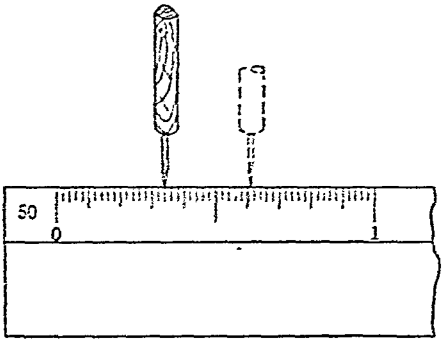


Fig. 9

Engineering students are taught the principles and techniques of drafting early in their career, and we shall add but a few suggestions on precision drafting before proceeding with the applications.

First, the equipment of T-squares and triangles should be tested for their accuracy before proceeding with graphical solutions. Tests for these instruments will be found in any of the textbooks or manuals of Engineering Drawing.

Next, the needle-point should be used in marking off all measurements to insure greater precision than is possible with a pencil-point. Drafting needle-points, consisting of a needle mounted in a wooden handle, are available at all drafting supply houses. The use of the needle-point is illustrated in Fig. 9. It will be noted that measurements to the nearest one one-hundredth of an inch may be made without difficulty through the use of the scale, commonly available, which is divided into fiftieths of an inch, and the needle-point to split these divisions into halves, yielding hundredths of an inch. In graphical solutions, the decimal system of inches with component parts reported as decimal fractions is standard practice. In using the needle-point, it is advisable to close one eye when marking a distance to avoid parallax.

No measurements should be made "in the air"—that is, by pricking holes at the ends of a measured distance, and then connecting these extremities with a line. Instead, measurements must always be made by first drawing a faint layout line, upon which the distance is measured. The layout line is made longer than the desired distance, the surplus line erased after measuring, and a sharp finish line applied exactly over the layout line. Since the needle-point leaves a tiny hole, the erasure of extra layout line leaves the extreme points intact to limit the finish stroke. When these operations have been completed, the scale should be applied to the line and the final length of line again measured as a check. The practise of measuring distances by setting the dividers on a wooden scale, and then transferring the distance with the dividers to the paper, should be condemned; there is involved a probability of inaccuracy in transfer which the needle-point system completely avoids, and, most important, the psychology of respect for the measured distance is disturbed.

In setting the compasses, the radius should be laid out on a drawn layout line, and the compasses set on the measured line. Precision drafting is a matter of correct techniques, not inborn skill, and, from the start, one should not be content with any devices but those

which will insure the highest degree of accuracy. The use of such phrases as "the highest degree of accuracy" and the pungent word "precision" need scare no one, since no native skill or special talents are required. Mechanical routines readily mastered are the basis of operations, and when treated with the respect they merit, insure precision.

All line-work, whether "layout" or "finish," must be extremely sharp, or precision is sacrificed. In this type of drafting the definition of Euclid—"a line has length but no breadth"—is a penetrating one.

Hard, sharp drafting pencils must be used. The usual system of instructing users that a "4 H" pencil should be used for this type of line and the "2 H" for that type is open to criticism. No two individuals have identical degrees of hand pressure on a drafting pencil, and the resulting lines yielded by the same pencil in the hands of different individuals may vary widely. The line itself should be used as guide, and each individual can determine the proper hardness of pencil which he must use to reach an objective of proper lines. Layout lines should be so faint that they can just be seen for measuring purposes; finish lines should be darker, but not broader. In both cases the lines must be as sharp as constant use of sandpaper block or file can produce.

This discourse upon the character of the pencil-point may seem unnecessarily insistent, but precision drafting rests upon the sharp line. The attitude of the user of sharp lines fosters a wholesome respect for the value of graphical solutions.

A further aid to extreme precision is the magnifying reading glass, mounted upon a small stand to leave the hands free, and used over the scale and needle-point when measuring. This device is not always essential, but guarantees, at little additional effort, even higher degrees of accuracy.

With proper equipment, techniques, and attitude, the accuracy of a graphical solution increases in direct proportion to the scale, or size of a drawing, all other factors (like the human one) being equal. It will be noted that if a line one inch long is to be measured to the nearest $\frac{1}{100}$ of an inch, a possible error of one one-hundredth of the measured length or 1 per cent is being tolerated. If we measure a 10-inch line, which can still be measured to the nearest $\frac{1}{100}$ of an inch, then the possible error which is tolerated becomes one one-thousandth of the measured length, or 0.1 per cent. It follows that as large a scale as is convenient should be used in all graphical solutions.

The qualifying phrase, "factors like the human one" should be noted. Increasing the scale of the drawing presents no inviolable guarantee of accuracy. The proper construction of parallels and perpendiculars, the obtaining of sharp intersections between lines, are equally influential in determining the degree of success, but in all cases, since the drawing is mechanical or instrumental in its routine, a mastery of correct drafting technique will insure proper results.

In measuring angles, the use of the usual small stamped protractors should be avoided, because inaccurate measurement of angles may nullify the care taken with all of the linear measurements. Such instruments are too crude for accurate measurement of angles.

The table of chords should be used, instead, and such a table with instructions for its use is given in the appendix. The tabulated values give the chord for any angle when the intercepting radius is of unit length. In measuring angles, a longer radius should be chosen and the tabulated value of chord multiplied by the ratio of the actual radius to the unit radius which forms the table basis.

Problems 1–2. In a system of vectors, all vectors are of the same inclination but are of different sense and magnitude. Vectors of one sense are labelled (+) and those of the opposite sense (−). Determine and report the resultant graphically.

1.

Vector	*Sense*	*Magnitude*
A	+	35
B	−	82
C	−	13
D	+	24
E	+	28

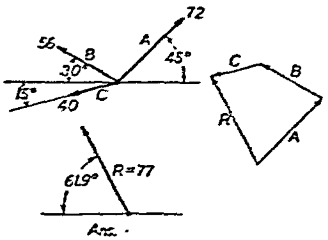
Ans. −8.

2.

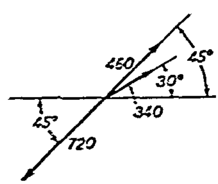
Vector	*Sense*	*Magnitude*
A	+	122
B	+	308
C	−	176
D	−	43

Problems 3–7. Given the system of vectors shown in the figures. Determine the resultant, graphically. In each problem check the resultant by using the commutative law, that is, by using a different order in adding the vectors. Report the magnitude, inclination, and sense of the resultant.

Problems 8–10. In the given system of vectors, the magnitudes, inclinations, and senses of vectors A, B, and C are known. If the inclination of vector D and of the resultant R are also known, find the magnitudes and senses of R and D.



PROB. 3



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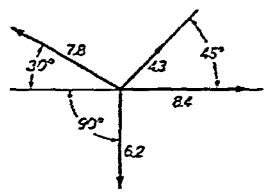

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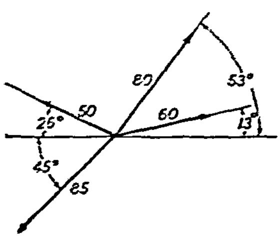

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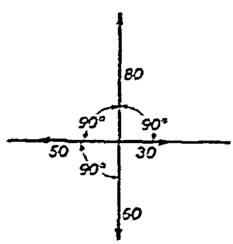

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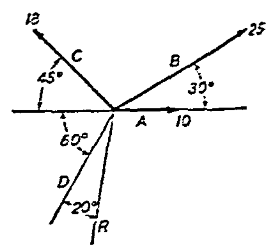

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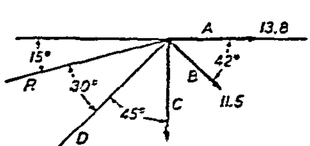

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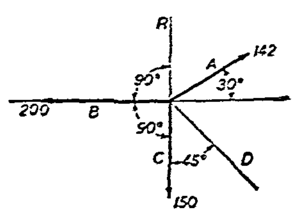


PROB. 10

6. Vector Addition by Parallelogram. To find the resultant of a system of two vectors, there is available the method of addition shown in Fig. 10. We are given two vectors, A and B, whose resultant is to be obtained. We must, then, add the vectors to find their combined effect.

A line from the terminus of each vector parallel to the other vector is drawn forming a parallelogram. The diagonal R (Fig. 10-a) is the resultant of the two vectors A and B. Note that since the geometric figure is a parallelogram, C is a vector having the same magnitude, inclination, and sense

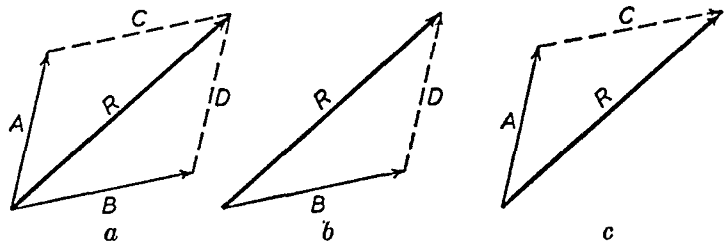
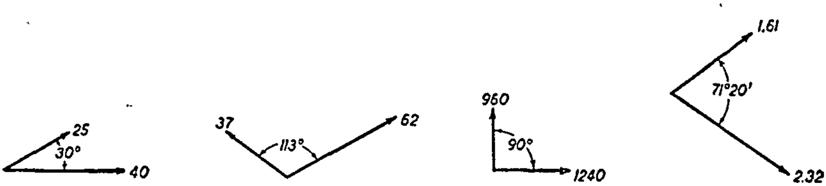


FIG. 10

as B, and D is equivalent to A. The resultant may be checked by adding B and D (Fig. 10-b), an equivalent system to B and A, or by adding A and C (Fig. 10-c), an equivalent system to A and B. The commutative law is authority for proceeding in either direction, and either yields the same resultant. The parallelogram of Fig. 10-a accomplishes the same purpose; it is the method most commonly used to obtain the resultant of a system of two vectors.

PROBLEMS

Problems 11–14. Given the pair of vectors shown in the figures. Find the resultant by the parallelogram method, graphically. Report magnitude and sense of the resultant, and the angle which it makes with the smaller vector.



PROB. 11 PROB. 12 PROB. 13 PROB. 14

7. Subtraction of Vectors. The study of algebra has presented us with the concepts of positive and negative quantities.

In analyses of vector quantities, it will be useful to adopt a similar conventional treatment of these opposites.

In algebra the negative of a number is expressed by placing the minus sign before the number. In analysis of vectors the negative of a vector is expressed by *reversing the sense* of the vector.

As an illustration, the vector of Fig. 11 represents a displacement of 10 ft. at an angle of 45° with the X-axis and having its sense upward to the right.

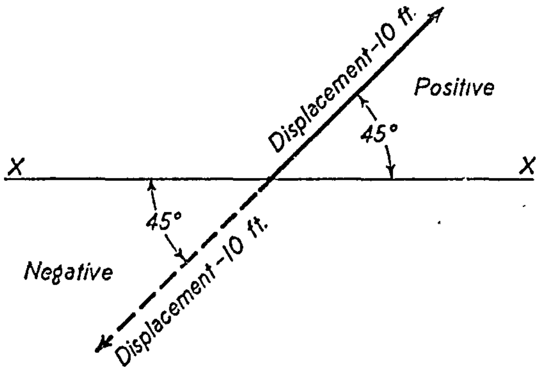


FIG. 11

The negative of this vector is shown and has the same magnitude and inclination as the first vector but opposite sense. If either vector is assigned the description positive, the other, which is of reverse sense, is its negative.

If in algebra

$$a - b = c$$

it is equally true that

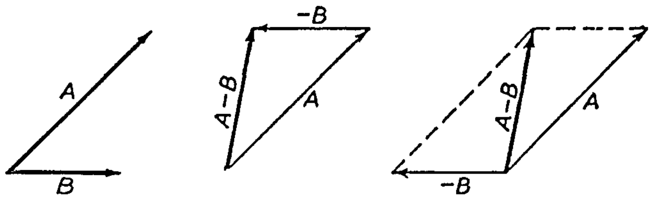
$$a + (-b) = c.$$

In the first example the direct process of subtraction is being used. In the second, the same purpose is accomplished and the same difference c obtained as the answer by using the process of addition, but we are adding to a a negative b.

In subtracting vectors, the process of addition of the negative vector is employed.

To find the difference between vector A and vector B, Fig. 12-a, we add to vector A the negative of vector B, which is $-B$, having the same magnitude and inclination as B but opposite sense. This addition, as before, may be made by adding the vectors directly, as in Fig. 12-b, or by parallelogram, as in Fig. 12-c.

The process may be confirmed by noting that if we add to one vector another of equal magnitude and inclination, but of opposite sense, we would find the terminus of the second vector at the origin of the first, since the



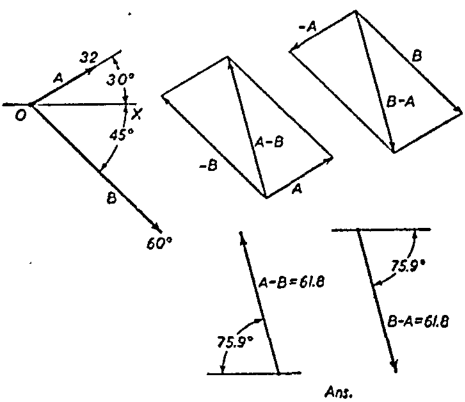
a b c

FIG. 12

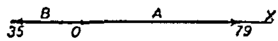
two vectors must, because of their nature, balance each other, and yield a resultant of zero magnitude.

PROBLEMS

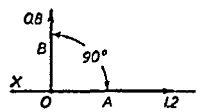
Problems 15–20. Given the vectors A and B. Find graphically
(a) the vector difference A–B.
(b) the vector difference B–A.
In each case, report the magnitude and sense of the vector difference and the angle which it makes with the X-axis.



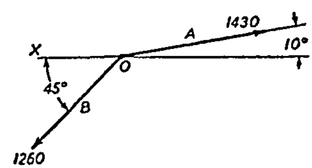
PROB. 15



PROB. 16



PROB. 17



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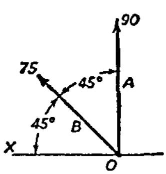

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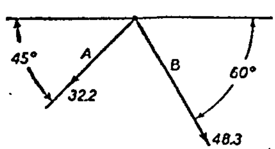


PROB. 20

8. Components of Vectors. Resolution. We have now combined groups or systems of vectors by adding to find the resultant, which is an equivalent but simpler system. Of equal importance in kinematics is the substitution of a system which may be more complex in that it substitutes for

the original single vector an equivalent system containing more parts. This type of substitution is called *resolution.*

A displacement of a body from point *1* to point *2* (Fig. 13) may be accomplished by forcing the body to travel directly along the path *1–2,* or by the indirect route *1–3–2,* or *1–4–5–2,* or by any other route that fancy or actual necessity may dictate.

Turning to vectors, we find that we may break up, or resolve, the resultant vector *1–2* by substituting for it a more indirect or complex group of displacement vectors, provided that, as substitutions are made, each replacement forms an equivalent system. The vectors which form the new

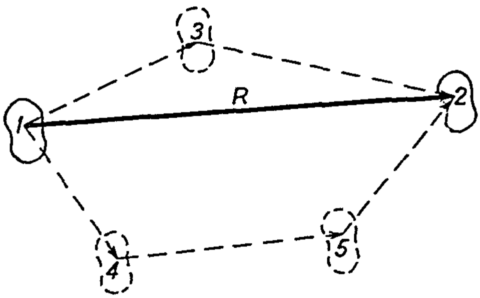


Fig. 13

group are called *components* of the resultant. In finding components of a given resultant, it is always important to check the solution to insure that proper components have been established. Proper components of any resultant are those which form an *equivalent* system. The equivalence muct be absolute; the components must, therefore, be of such magnitude, inclination, and sense that, when added, they yield the original resultant. Any resultant has an infinite number of possible components. We shall find, however, that in kinematic analyses, the resultant is usually resolved into an equivalent system of two components.

Starting, as in Fig. 14, with a resultant vector R, its components in the directions of the two reference axes S–S and T–T are to be found.

Starting at the terminus of R, a line parallel to S–S is drawn to meet axis T–T. We now have two component vectors B and C, each of the desired inclination and of magnitude which is now determined. To determine the sense of the components we must note that their sum must proceed from the origin of R to its terminus.

A trial of the two possible senses of B and the two of C reveals that the senses shown in the figure must be assigned, since these are the two components which, when added, give the original vector R as their sum, or resultant.

While the method of analysis shown in Fig. 14 has been chosen to afford a clear illustration of process, the parallelogram method is most widely used in applications.

Figure 15 shows the construction of the parallelogram which accomplishes

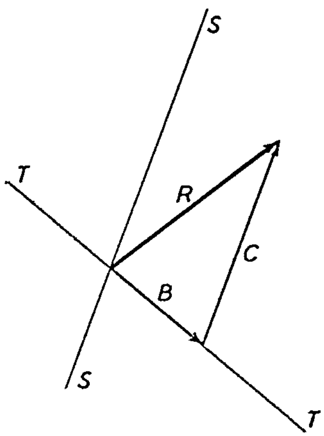


Fig. 14

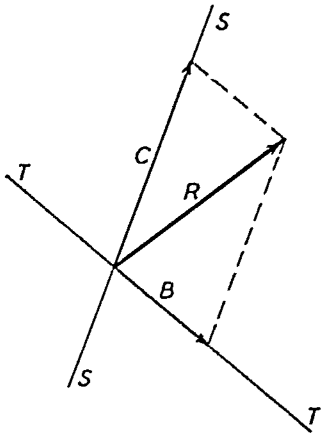


Fig. 15

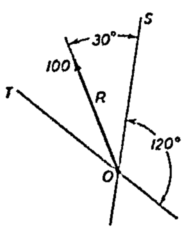
the same purpose of resolving vector R into an equivalent system consisting of vectors B and C.

In either of the methods the result is checked by noting that the component vectors B and C, if added, give a sum, or resultant, R.

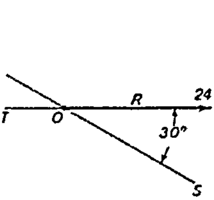
Any resultant is its own component upon an axis parallel to it, and has zero component upon an axis at right angles to it.

PROBLEMS

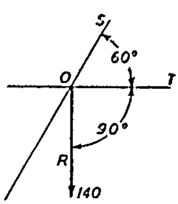
Problems 21–24. The vector, R, is given. Resolve R into its components along the S- and T-axes, graphically, and report the magnitude and sense of the components.



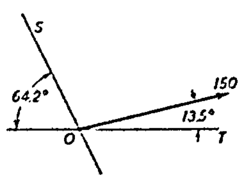
Prob. 21



Prob. 22



Prob. 23



Prob. 24

Problems 25–27. The given vector, R, is to be resolved into S- and T-components R_S and R_T, respectively. The angle between R and the T-axis is given, and the ratio of the magnitudes of the components, $\frac{R_S}{R_T}$, is given. Find R_S and R_T.

25. This problem illustrates the possibility of establishing a background from an analytical study for a method of graphical attack.

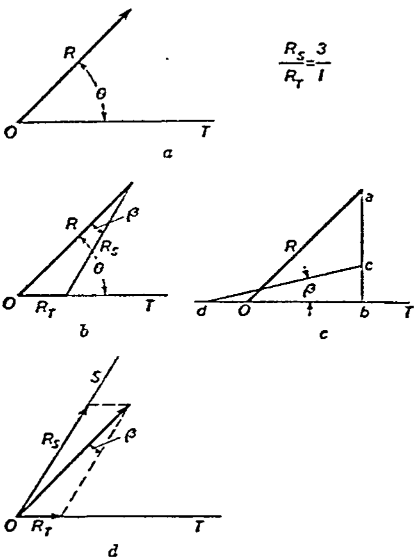
In 25-a, R and θ and the ratio between R_S and R_T are known. Let x (25-b) be the magnitude of the T-component, R_T. Then $3\,x$ is the magnitude of the S-component, R_S

$$\frac{\sin\beta}{\sin\theta} = \frac{x}{3\,x} = \frac{1}{3}$$

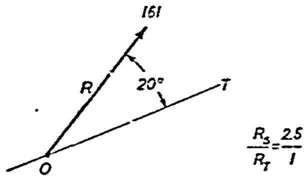
$$\sin\beta = \tfrac{1}{3}\sin\theta$$

If, as in 25-c, a perpendicular ab is dropped from a, the terminus of R, to meet the T-axis at b, the sine of θ is $\frac{ab}{R}\cdot$ bc is laid off equal to $\frac{1}{3}ab$. From point c, a radius equal to R is swung to intersect the T-axis at d. Then the angle $cdb = \beta$.

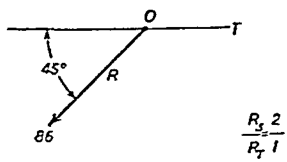
Now as in 25-d, the S-axis may be drawn, making an angle β with the resultant, and R_S and R_T have been determined.



PROB. 25



PROB. 26

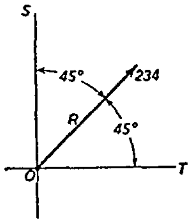


PROB. 27

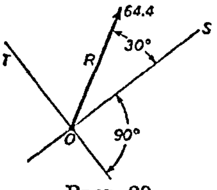
9. Rectangular Components. When the two axes S–S and T–T of Art. 8 are at right angles, the parallelogram of resolution becomes a rectangle. The components B and C are at right angles to each other and are called *rectangular components*.

PROBLEMS

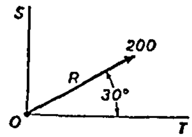
Problems 28–33. The given resultant is to be resolved, graphically, into rectangular components along the S- and T-axes, which are perpendicular to each other.



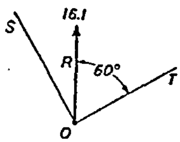
PROB. 28



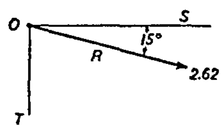
PROB. 29



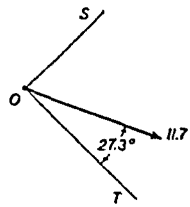
PROB. 30



PROB. 31



PROB. 32



PROB. 33

10. Analytical Method for Finding the Resultant. The graphical method which has been employed in the analyses of vectors is direct and efficient.

The method of attack on vector quantities may also be an analytical one. While, in general, applications in the field of mechanisms are made graphically, a discussion of the equivalent analytical methods will serve both as an additional tool, and to strengthen appreciation of the principles underlying the geometry of the graphical constructions.

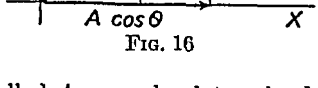


FIG. 16

A resultant has already been defined as the simplest equivalent of the original system of vectors.

If a vector like A of Fig. 16 be known, its component in the X-direction, which will be called A_x, may be determined.

$$A_x = A \cos \theta$$

and its Y-component

$$A_y = A \sin \theta.$$

In a system of vectors, like A, B, C of Fig. 17, each vector will have an X-component, and each will have a Y-component.

e have now reduced the original system to an equivalent system of two ctors, ΣX and ΣY, which are shown in Fig. 18.

This pair may be further reduced by finding their vector sum. Since X and ΣY are at right angles, their vector sum will be

$$
\begin{aligned}
R &= \sqrt{\overline{\Sigma X}^2 + \overline{\Sigma Y}^2} \\
&= \sqrt{\overline{47.42}^2 + \overline{28.61}^2} \\
&= 55.38
\end{aligned}
$$

Here we have used the Pythagorean theorem, and have noted that the quare of the hypothenuse of a right triangle is equal to the sum of the quares of the two other sides. We have thus paralleled, by an analogous oute, the graphical method of addition.

The inclination of the resultant may be found, for it will be noted that he angle between the resultant and the X-axis is

$$
\begin{aligned}
\theta &= \tan^{-1} \frac{\Sigma Y}{\Sigma X} \\
&= \tan^{-1} \left[\frac{28.61}{47.42} = 0.6033 \right] \\
&= 31.1^\circ
\end{aligned}
$$

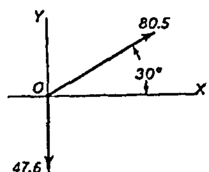
The sense of the resultant is established from Fig. 18 by noting the enses of the X- and Y-components.

The final resultant, the simplest system which is equivalent to the riginal one, is shown in Fig. 19.

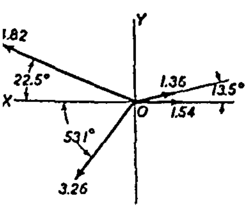
PROBLEMS

Problems 34–37. Given the system of vector quantities shown n the figures. Determine the resultant, analytically. The X- and Y-axes are mutually perpendicular.

Ans. to Prob. 37. $R = 13.9$ at $\theta = 8.7^\circ$



PROB. 34



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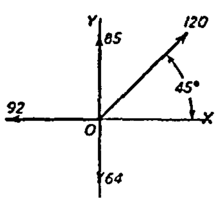

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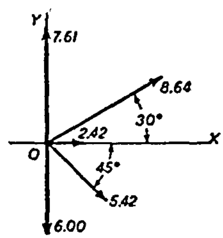


PROB. 37

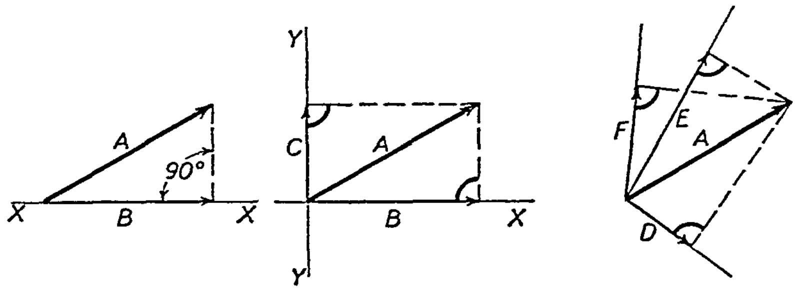
38. Solve Problem 30 analytically. *Ans.* $R_S = +100$; $R_T = 173.2$
39. Solve Problem 31 analytically.
40. Solve Problem 32 analytically. *Ans.* $R_S = +2.53$; $R_T = -0.68$
41. Solve Problem 33 analytically.

11. Orthogonal Components. The vectors which have thus far been discussed have been of general interest and of universal application as representations of vector quantities.

In the study of kinematics, there is a class of component vectors which are of particular interest, and which will therefore be segregated for special attention.

Rectangular components have already been defined as those obtained by resolution when the axes of the components are at right angles.

In many applications, both of the rectangular components are constantly employed in analysis. In our studies of velocity and acceleration, we shall



a b c

Fig. 20

find that constant use may be made of one of the mates of a pair of rectangular components. While the complementary or mated component is always present, an individual rectangular component may, by itself, become a very effective means of attack. Since we shall use such an isolated component constantly, it will be convenient to give it a distinguishing name, the *orthogonal* (right-angled) *component.*

An orthogonal component is obtained by projecting any vector upon a desired axis. The projection is orthographic projection, that is, the projector is always perpendicular to the axis.

In Fig. 20-a, vector B is the orthogonal component of vector A on the axis X–X. Vector C, shown in Fig. 20-b, is the mated rectangular component of B, and is therefore the orthogonal component of A on the axis Y–Y. Other typical orthogonal components of A are D, E, and F which are illustrated in Fig. 20-c. In each case, the heavy arc represents an angle of 90 degrees.

12. Theorems of Orthogonal Components. From the definition of orthogonal components, it will follow that there are many vectors like A, as A_1, A_2, A_3, etc. (Fig. 21) which, since they have a common origin O, will have the same orthogonal component B along the X-axis. Each of these possible vectors may be resolved into a pair of rectangular components, as B and C_1, B and C_2, B and C_3.

This leads to a related thought. If it is known that the orthogonal component of a resultant vector along the axis X–X is B, the resultant vector must have an origin O, and a terminus which lies somewhere in the line *1–2* perpendicular to the X–X axis at the terminus of B (Fig. 22).

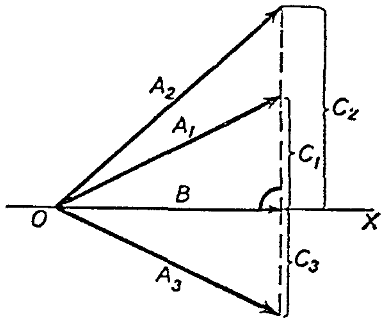


FIG. 21

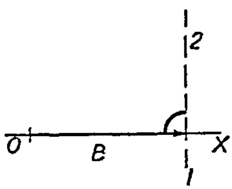


FIG. 22

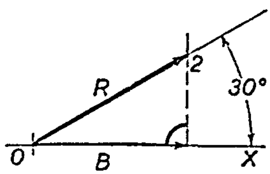


FIG. 23

Now if, in addition to orthogonal component B, the inclination of the resultant is known, that resultant may be determined.

Given an orthogonal component B in Fig. 23. B's inclination is selected as an X-axis.

Given also the inclination (as 30° with the X-axis) of a resultant vector R of which B is an orthogonal component.

To find the resultant we construct a line making an angle of 30° with the X-axis from origin O. The resultant has now been fixed at its proper inclination. Next, a perpendicular to X–X at the terminus of B is erected, and this perpendicular is extended to meet the line along which R lies at point *2*. Then $O2$ is the magnitude of the resultant R. R must have a sense upward to the right as shown in order that the given sense of its orthogonal component B shall be correct.

This development is summarized as:

***Theorem I.** One orthogonal component and the inclination of the resultant vector determine the resultant vector.*

Complete knowledge of a resultant vector may be built up from another source of information.

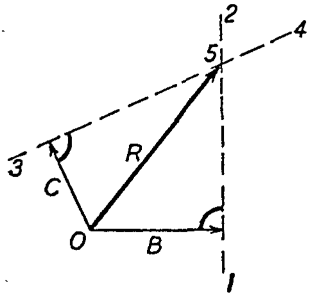


Fig. 24

Two orthogonal components of the same resultant are known as B and C in Fig. 24.

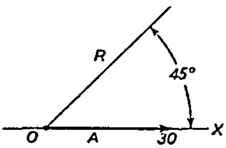
Any resultant vector of which B is an orthogonal component must have an origin at O and a terminus which lies in the line *1–2*.

Similarly, any resultant vector of which C is an orthogonal component must have an origin at O and a terminus in the line *3–4*. Point *5* is the only point which can lie in both *1–2* and *3–4* and is, therefore, the terminus of the resultant vector R, which is now fully determined. This information may be summarized as:

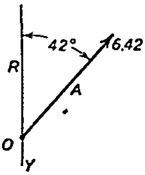
Theorem II. Two orthogonal components of any vector determine the resultant vector.

PROBLEMS

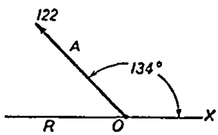
Problems 42–45. Given one orthogonal component, A, and the inclination of the resultant, R. Determine the magnitude and sense of the resultant.



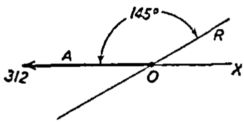
Prob. 42



Prob. 43

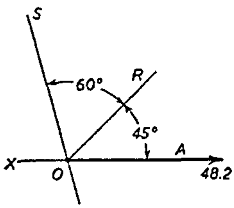


Prob. 44

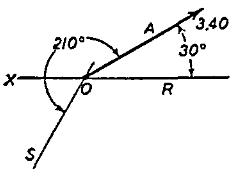


Prob. 45

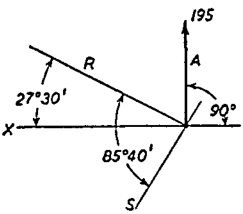
Problems 46–49. The given data consists of a known orthogonal component, A, the inclination from A of the resultant, R, and the inclination of axis S. Find the orthogonal component of R along S, graphically.



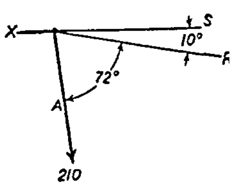
Prob. 46



Prob. 47

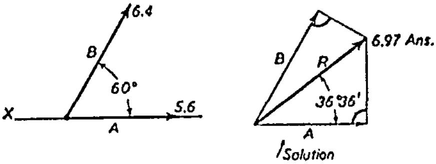


Prob. 48

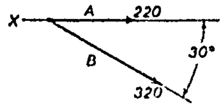


Prob. 49

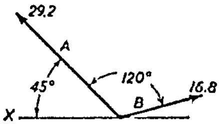
Problems 50–54. Two orthogonal components, A and B, are given. Find the resultant vector, R, graphically.



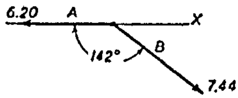
PROB. 50



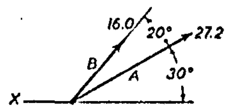
PROB. 51



PROB. 52

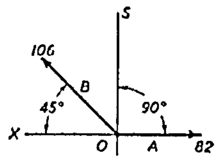


PROB. 53

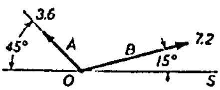


PROB. 54

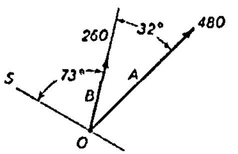
Problems 55–58. Two orthogonal components, A and B, are known. Find, graphically, the orthogonal component C, which is the orthogonal component of the resultant of A and B along the S-axis.



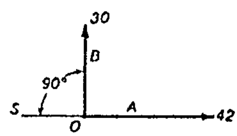
PROB. 55



PROB. 56

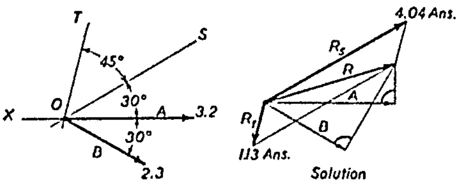


PROB. 57

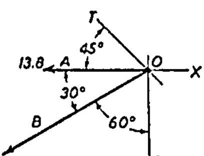


PROB. 58

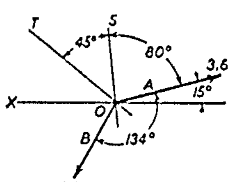
Problems 59–62. The distinction between the "ordinary" component vector, based upon the parallelogram relationship, and the special "orthogonal" component vector should be well defined. Two orthogonal components, A and B, are given. Find, graphically, the S- and T-components of the resultant of A and B. When the term "component" is used, ordinary component is intended; when orthogonal components are to be determined, they will be so qualified.



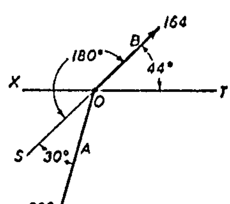
PROB. 59



PROB. 60



PROB. 61



PROB. 62

13. "Mated" Orthogonal Components. Before leaving these theorems to analyze the application of vectors in kinematics, it will be well to note again that there is associated with every orthogonal component a mate which is the second member of a pair of rectangular components. In the illustration used in Fig. 24, where a resultant has been determined through knowledge of two orthogonal components, we may find (see Fig. 25) B', which together with B forms a system of rectangular components. These two, B and B', if added will yield as their resultant the vector R.

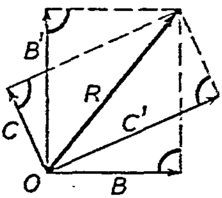


Fig. 25

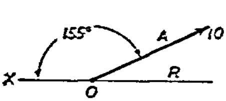
Orthogonal component C has as its rectangular mate the vector C'. This pair also gives the resultant R when added.

Systems B and C, B and B', C and C', B and C', and C and B' are all equivalent systems of orthogonal components since each system yields the same resultant.

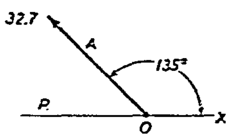
Any two of these orthogonal components will determine the resultant R, for they satisfy the demands of Theorem II.

PROBLEMS

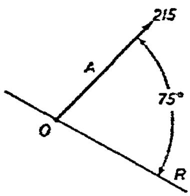
Problems 63–66. The inclination of the resultant, R, with a known orthogonal component, A, is given. Find the "mated" orthogonal component, B, which is the orthogonal component of R perpendicular to A, graphically.



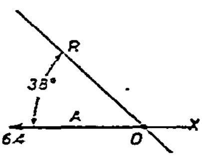
Prob. 63



Prob. 64



Prob. 65



Prob. 66

IV

MOTION

14. Kinematics is the branch of mechanics which investigates the motion of bodies, without accompanying analyses of the forces which must be acting to produce the motion, or of the mass properties of the bodies themselves. It is then a study of the geometry of motion, and of the time element, as fundamental concepts, and of velocity and acceleration which are derived from these fundamentals.

An effective approach to the subject of kinetics, which merges the study of force system, mass properties, and accelerations, demands full appreciation of the kinematics involved.

In addition, the design of mechanisms to accomplish desired motions rests upon the application of the principles of kinematics and forms the subject of applied or *engineering kinematics.*

15. Mechanisms and Machines. Mankind has produced countless mechanical devices of wide range and great variety, and of remarkable ingenuity. Through the ages applications of mechanical substitutes for manual skill have poured into the stream of society, until society accepts them as nonchalantly as though they had existed forever.

Inventors have dreamed, schemed, and wrought during every historical era, and it would appear that any exploration of mechanical movements would require long, painstaking study and voluminous reports of description.

However, just as a coral reef is built by myriads of microscopic animals in succeeding generations which deposit their skeletons in turn on those of their ancestors, the processes of invention and of development of applications of motion rest in each stage upon their antecedents. Fundamental forms of mechanism serve as a basis of mechanical progress, and new forms branch from the main trunks. Each new application of motion derives from a parent form, and ramifies or amplifies the heritage which it has received.

In the study of engineering kinematics, our primary concern lies in the roots or bases, rather than with ramifications.

The common terminology of mechanisms, and the basic elements of mechanism must be explored to establish both a working vocabulary and an appreciation of the general or broad aspects of applied motion, which may serve as adequate preparation for more detailed investigation.

A mechanism is a combination of two or more rigid bodies, so arranged that imparting motion to one body compels the remaining bodies to have specific motion.

The body to which the original motion is imparted is called the *driver*, and the final body of the remaining group comprising the mechanism is called the *follower.*

The driver and follower may be directly in contact with each other, as in the case of the pair of gears shown in Fig. 26.

Here we note that if the driver is turned, its teeth compel the follower to turn.

Frequently, other bodies are interposed between the driver and follower. In Fig. 27, a series or train of bodies is used in transmitting motion from

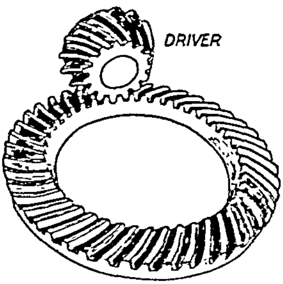


FIG. 26

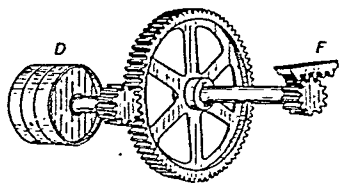


FIG. 27

the driver D, which is a wheel, to the follower, F, which is a sliding rack, constrained so that it moves in fixed guides.

In the mechanism descriptions of this book, the symbol D will represent the driver, and F the follower.

A machine is defined as a combination of two or more resistant bodies, constrained so that they move relative to each other, and also transmit force from the source of power at the driving source to the work which is to be done.

We note, then, that the term machine includes the concept of mechanism, and also embraces a function of work or energy modification. There is, then, distinction between mechanism and machine, and we shall use the term *mechanism* in its exclusive sense of a motion modifier, and preclude any analysis of the forces which are acting upon the bodies involved in producing the motion.

In establishing this concept of the mechanism, it will be observed that the definition is broad enough to include the instrument. The distinction between the two terms is not definite; it should not be, for any instrument which is based upon mechanical movements is, as such, a device employing

the same driver-follower relationships as any other mechanism. We shall have as our objective an appreciative understanding of general principles which are capable of universal application to all mechanisms. We shall not have occasion to confine our interest to the instrument, and shall therefore not segregate it for special treatment. An instrument is a mechanism designed primarily for accuracy. When used as instruments, mechanisms are generally small, the magnitude of external work done is negligible, and design for stress considerations is rarely necessary. These mechanisms form a subdivision of the general applications, in which accuracy is of fundamental importance. They do not differ in other respects, and their design and analysis will comprise the same kinematic principles as all other mechanisms.

The common elements of mechanism are cams, rolling-contact bodies, flexible connectors, gears, screws, and rigid-connectors, or link-work.

16. Cams. A *cam* is a body whose surface is designed so that it transmits a desired form of motion to another body, called the cam follower. This

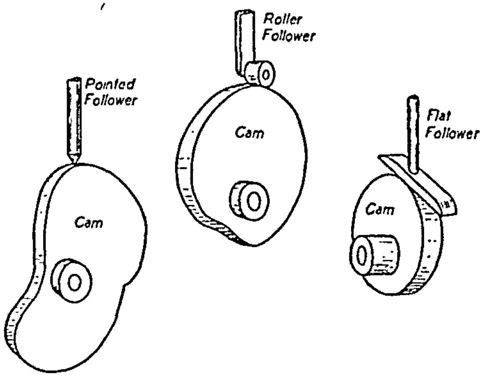


FIG. 28

definition is sufficiently broad so that it becomes a general description of the relation between most drivers and followers, particularly those in direct contact. In engineering practise, however, the term is reserved for a plate or cylinder whose action may best be illustrated through the following examples.

Figure 28 illustrates a group of plates whose surface is so shaped that rotation of the plate compels the follower, which is in contact with the surface, to have a definite form of motion. A spring or other restraint causes the surface of the follower to remain at all times in contact with the plate cam, or, as in the case of the positive motion cam shown in Fig. 29, a positive

motion of the follower is accomplished through such devices as cutting a groove, and the follower's entire cycle of motion is positively controlled by the cam, without recourse to springs or other auxiliary equipment.

If we note that any modification of the surface in the case of the cams of Fig. 28 or the groove of Fig. 29 immediately changes the character of the

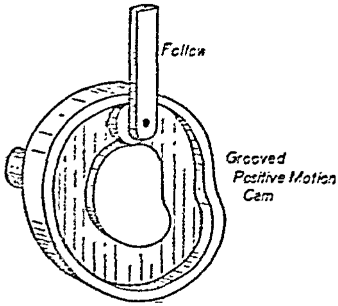


FIG. 29

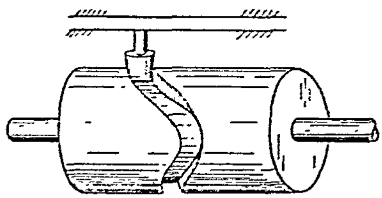


FIG. 30

follower's motion, we may appreciate the flexibility and versatility of this element of mechanism.

The plate cams which have been illustrated act upon followers which, in general, are constrained to move in planes perpendicular to the axis of rotation of the cam.

Cylindrical cams, as their name implies, consist of cylinders or barrels having grooves upon their surface, into which the followers fit, as in the illustration of Fig. 30.

The follower of a cylindrical cam moves in a plane containing the axis of the cam itself.

A single cam is frequently used, as in opening the valve of an automobile

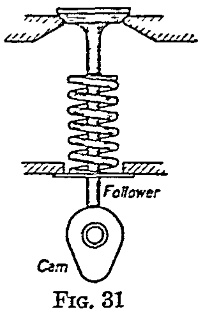


FIG. 31

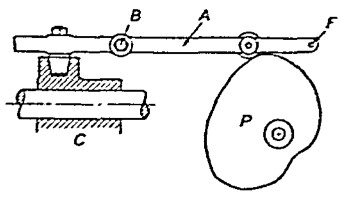


FIG. 32

against the action of a spring (Fig. 31), or cams may be used in combination, as in Fig. 32.

In this arrangement, the follower is point F.

Point F is to have resultant motion which, while remaining in one plane, has both horizontal and vertical components, that is, point F is to be moved either left or right, and, simultaneously, up or down.

The arm A is pivoted at B. The cylindrical cam, C, will supply the horizontal component motion of the pivot, and the plate cam P will act upon the arm to supply the vertical component motion.

Such combined motions may be successive or simultaneous, continuous or intermittent. Since the location of a point in a plane is determined by two coordinates, such a pair of cams make it possible to transfer the follower's point, F, to any desired position in the plane.

The procedure might be carried one step further. If the framework supporting the pair of cams as a unit is mounted upon a movable carriage, a third cam may then be introduced to compel that carriage itself to be moved in the remaining coordinate direction and the point of the follower, given three coordinate motions, can be transferred anywhere in the space limited by the dimensions of the machine.

The ability of cams to guide a follower over any desired path is comparable to manual skill. In machines where motion is so complex that we seek for a substitute for the flexibility and dexterity of the human hand (a substitute which will not tire, and which will repeat continuously and accurately the same path of travel) these elements of mechanism may be employed.

The development of many forms of modern automatic machinery has been made possible through the ingenuity of designers who took full advantage of the innumerable forms of motion made possible by properly designed cams.

The details of design of cams, and the methods of analyzing their motion and the motion of their followers—their displacement, velocity, and acceleration—will be developed as the general discussion of those properties is amplified.

17. Direct-Contact Bodies. Like the cam and cam-follower motion, which depends upon the contact of properly shaped surfaces, other driver-follower mechanisms which are dependent for their action upon direct surface contact are encountered.

One group or family of such mechanisms comprises the rolling-contact or friction-wheel types. In each of these, the frictional resistance of the surfaces in contact compels the drive. The surfaces are mounted so that they are pressed together, and are faced with yielding materials, like leather, wood, or rubber, to prevent slipping. By yielding the line of contact is changed to a surface of contact, and the resistance to slipping is increased.

Typical of the group is the pair of rolling cylinders shown in Fig. 33. Rotation of one cylinder causes the second cylinder to rotate. If there is no slipping, that is, if contacting lines on the two cylinders remain together during the short time of contact, the motion is described as *pure rolling contact.*

If contacting lines drawn upon each body slide relative to each other as they meet, the pure rolling contact has been disturbed, and a constant speed of the wheel used as driver will produce a variable speed of the follower as the slipping occurs. The use of such bodies is confined, therefore, to cases where the friction between the cylinders may be relied upon to prevent

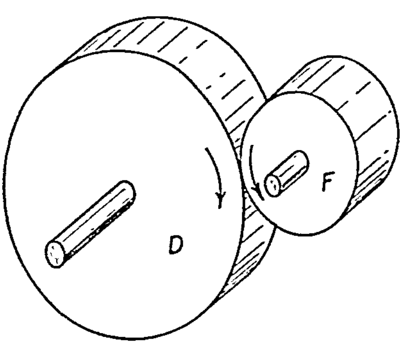


Fig. 33

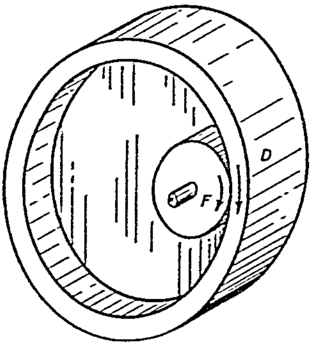


Fig. 34

slipping or where a small amount of slip may be tolerated. In machines which may receive shocks from suddenly applied abnormal loads, we frequently find intentional provision of the ability for parts to slip upon each other to relieve the danger of over-stressing the machine parts. In normal service, the impending slip limits the applications in which rolling cylinders can serve effectively.

These elements of mechanism, quite apart from their practical application, have very important significance, for as we engage in the study of the kinematic properties of rolling bodies we shall be laying the foundations for a very penetrating approach to the study of toothed wheels, or gears.

Rolling cylinders are employed when the shafts which form the axes of driver and follower are parallel.

When these shafts are to rotate in opposite directions the cylinders are mounted in external contact, that is, outside of each other as indicated in Fig. 33 (here a clockwise direction of rotation of D will cause a counterclockwise rotation of F).

If a drive is to be so arranged that the parallel driver and follower shafts are to turn in the same direction, the cylinders may be designed for internal contact as shown in Fig. 34 which consists of a small wheel placed within a

larger outer wheel, called the *annular;* or, as in Fig. 35 an intermediate wheel, called an *idler*, is inserted to provide the proper direction of rotation of the follower.

In both these cases the speed ratio is dependent upon the diameters of

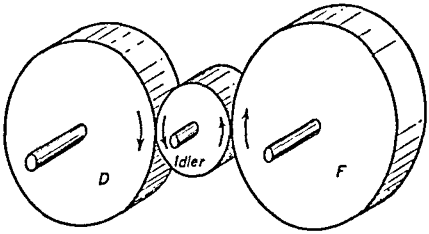


FIG. 35

the wheels, as we shall observe in our study of velocity. Since these diameters are constant, a pair or series of rolling cylinders will have a constant speed ratio, within the limit imposed by the beginning of slip.

When the shafts of driver and follower are not parallel, but intersecting, the rolling bodies which are used to produce the pure rolling contact are

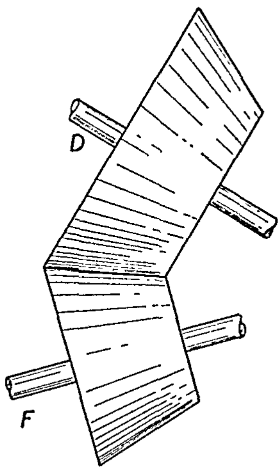


FIG. 36

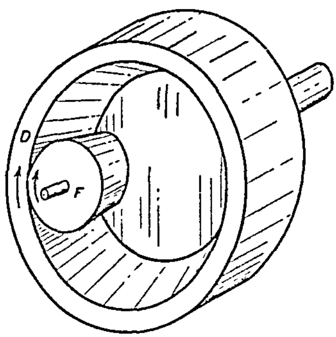


FIG. 37

conical. Pairs of *rolling cones* in both external and internal contact are shown in Figs. 36 and 37 respectively. These, too, have a constant speed ratio.

When a variation of the speed ratio is to be made available during operation, bodies like those shown in Fig. 38 may be used.

Figure 38 shows a pair of cones between which an endless leather belt is placed. The belt is held in position by an arm, or "shipper," which can shift the belt to any desired position and hold it there. In each position a new speed ratio is established which is dependent upon the diameters of those elements or sections in contact. This arrangement provides the opportunity of giving the follower F one of a large number of different speeds while the driver D continues to rotate at one constant speed.

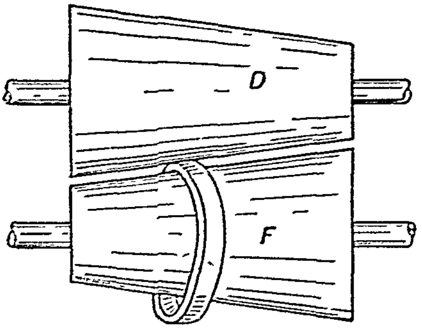


FIG. 38

Another mechanism, which is effective when changes of speed are vital, is the *disc-and-wheel* combination of Fig. 39. In this mechanism, friction between the large disc and the small roller is again employed to compel motion.

The speed ratio is dependent upon the distance x, so that if the roller is advanced toward the axis of D's shaft or retarded from it, differing speed ratios are established.

If the roller is moved so that it occupies a position such as F' on the opposite side of the disc's axis from its previous position, it reverses its direction of rotation. Such a mechanism presents, therefore, not only a change-speed device but the basis of a reverse-direction drive.

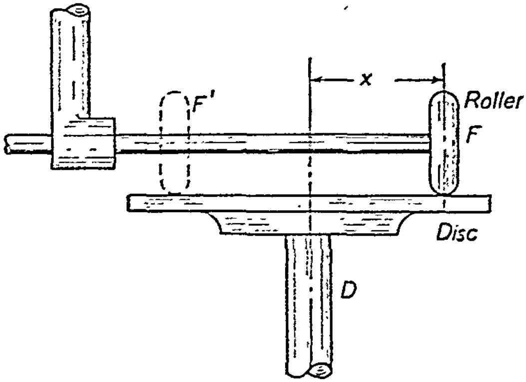


FIG. 39

A further development of the disc-and-wheel friction drive is illustrated in Fig. 40. In this mechanism two rollers are placed between hollow discs D and F. These rollers are supported so that they may be rotated about axes a (perpendicular to the plane of the drawing). The rollers are operated by a yoke or arm not shown in the figure, and changes of the speed ratio

between driver and follower are accomplished by moving the arm to change angle θ.

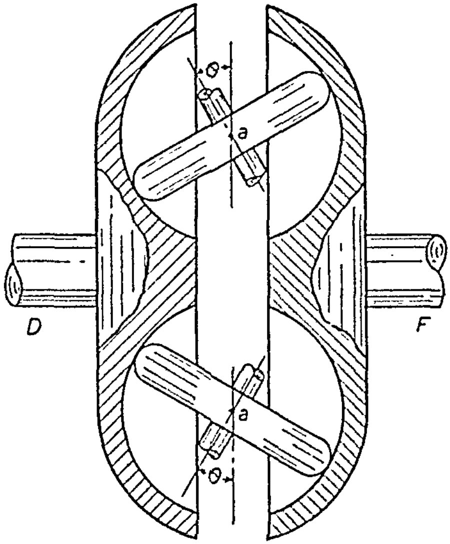


Fig. 40

18. Flexible Connectors. When the distance between the axes of driver and follower is too great for such direct-contact drives as those already discussed, or for gearing, to be used, flexible connectors, such as belts, ropes, or chains are employed.

Like friction wheels, these flexible connectors depend upon friction between surfaces to maintain the drive and are subject to similar limits imposed by slip.

Figure 41 illustrates an *open belt* drive. In this case both wheels or pulleys have the same direction of rotation. If the belt is mounted upon the pulleys in *crossed belt* arrangement, as shown in Fig. 42, the pulleys will have opposite directions of rotation. In either case, the speed ratio is constant, since it depends upon the diameters of the pulleys, which for any given pair are fixed.

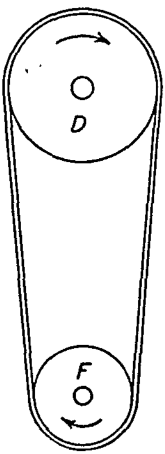


Fig. 41

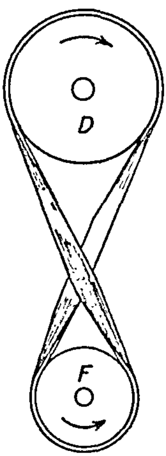


Fig. 42

Selectivity of speed ratio with flexible connectors has been introduced in many forms. The drive shown in Fig. 43 is a typical change-speed one.

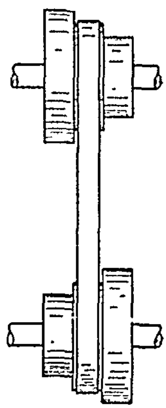


Fig. 43

Here a pair of pulleys, each of which has a series of diameters or steps, are connected by a belt. When the belt is in position upon any one of the matched pairs of diameters, as in the figure, a definite speed ratio of the pulleys is established. If the belt is shifted to one of the other steps, a different speed ratio is available for the drive. Another change-speed device

and the drive need not be stopped for repairs until such time as is convenient.

Chains are used for drives where the distance between axes is not great, and where a more positive drive than may be obtained through the friction transmission of belting is essential. Since they are all-metallic, they may be used where excessive heat or corrosive agents like oils would bar the usual forms of belting, particularly those of rubber. Several forms are employed. The "roller" and "silent-drive" types are shown in Fig. 46.

19. Gearing. The rolling cylinders and cones which have been described form a most simple type of drive. Simplicity is, in mechanism design as in other fields, a goal to be sought.

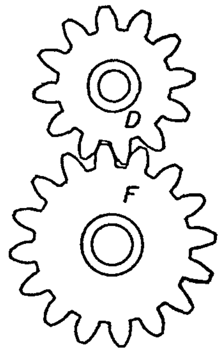
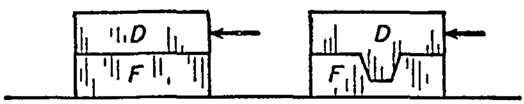


Fig. 47

The simplicity of such action would form the ideal drive were it not for the disturbing factor of slip, which, as has already been noted, disturbs the positive nature of such transmission of motion.

Very early in man's use of mechanical tools he discovered a means of improving the simple rolling cylinder or cone through adding teeth to the previously rolling surfaces. The drive was thus relieved of its reliance upon frictional resistance, and the action transferred to a dependence upon the shape of contacting surfaces.

In the drawing of Fig. 47 it will be observed that a tooth of the driver compels the contacting tooth of the follower to move, not because of



a b

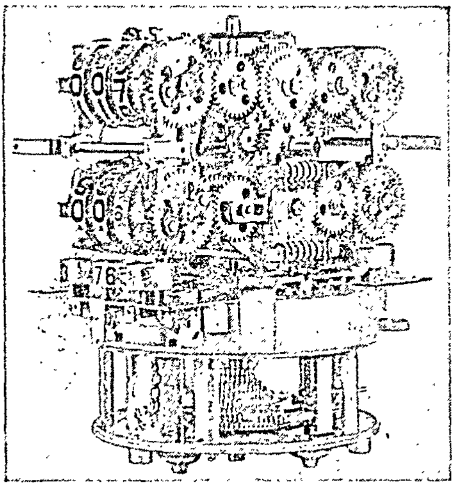
Fig. 48

friction, but through the property of shape. The distinction becomes more evident as the action of the two blocks shown in Fig. 48 is contrasted.

If the upper block of Fig. 48-a is rested upon the lower block and forced to move horizontally, the lower block is compelled to move only when the friction between the two blocks is sufficient to prevent relative motion between them, or slipping.

In the case of the upper block of Fig. 48-b which has a projection or

tooth on its lower surface fitting into a space cut in the lower block, a horizontal push on the upper block forces the lower block to move, and the drive becomes a positive one, limited only by the strength of the projecting tooth or the sides of the slot.



Courtesy Veeder-Root, Inc.
FIG. 49

The introduction of the toothed surface, on a gear, while making the drive positive, also introduces problems of the proper contour or profile to be given the contacting teeth.

The driving tooth of a gear operates in much the same fashion as a cam upon its follower, and the shape of its surface will determine the resulting motion of the follower. The surface must be carefully designed so that the resulting motion may be the desired one.

In addition to cylinders, cones or other rolling bodies may be modified by cutting teeth on their surfaces. Gears may be used in pairs, or pairs of gears combined in series or trains, as shown in Fig. 49.

20. Screws. An inclined plane may be used, as shown in Fig. 50, as a driver which will cause motion of a follower, as the inclined plane or wedge is moved.

If we visualize the inclined plane driver made of a thin sheet which is wrapped about a cylinder, as in Fig. 51, we have the screw, which compels

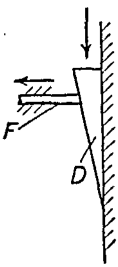


FIG. 50

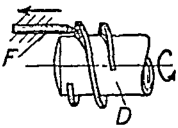


FIG. 51

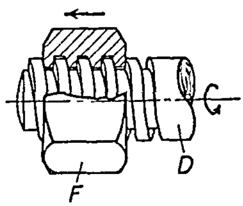


FIG. 52

the follower to move as the cylinder carrying the wrapped-on sheet revolves. The screw in finished appearance appears, as in Fig. 52, where each thread cut in the surface is serving in similar fashion to an inclined plane wrapped

on the cylindrical bolt, and the pointed follower has been further developed into a nut.

21. Rigid-Connector Mechanisms. Link-Work. A wide field of mechanisms is based upon series of rigid bodies, or links, used to transmit motion between driver and follower.

The basic mechanism of this class is composed of four bars or links, pinned

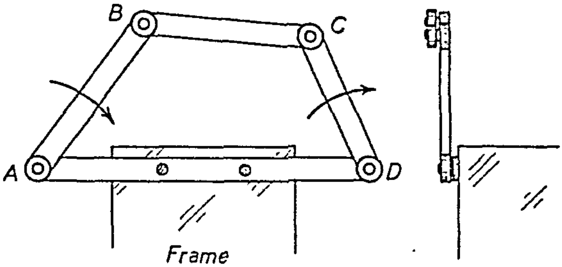


FIG. 53

together so that each pin acts as an axis and permits the two links which it joins to move freely relative to each other.

Such a four-bar linkage is shown in Fig. 53 where bars *AB*, *BC*, *CD*, and *AD* have been pinned together. In this case the bar *AD* is fastened to a stationary framework, and becomes the stationary link of the mechanism.

If *AB* is made a driver, any rotary motion given *AB* compels the follower, link *CD*, to move, the motion being transmitted through the link *BC* which serves as a connector.

By varying the dimensions of the four links relative to each other, a wide variety of kinds of rotary motion may be given the follower from a constant speed driver.

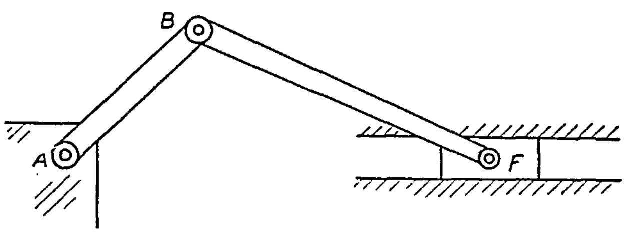


FIG. 54

Other forms of link-work than the pin-connected type are used; for example, in the case of the mechanism shown in Fig. 54, a link *AB*, used as driver, gives motion to follower *F*, which is a moving block, constrained to move in the fixed guides.

In this form, link-work is used in such a machine as a pump (see Fig. 55). Now the driver, AB, is a rotating member, called a crank, which drives a piston F (equivalent as far as motion is concerned to the sliding-block of the previous example, Fig. 54) through a connecting-rod BC. Or, as the basic linkage of the automobile engine, the piston D is made the driver and the crank becomes the follower, as in Fig. 56.

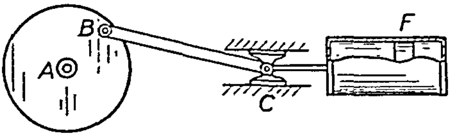


Fig. 55

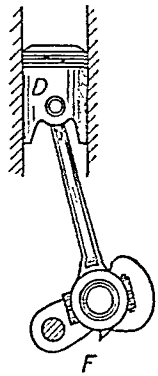


Fig. 56

Other mechanisms make use of series or trains of linkages. The mechanism shown in Fig. 57 is a quick-return mechanism which is found in such machines as tools used for cutting and fashioning the shape of metal parts. In this case, a cutting tool is attached to the sliding-block which serves as follower. In these mechanisms, a constant speed of the driving crank D causes the follower carrying the tool to have a fairly slow and uniform motion during the greater part of its cycle of operation. This portion is made the cutting stroke. The follower then returns very rapidly to its initial position, and the tool is ready to start upon another cut. Since the machine must be driven by a motor rotating at constant speed, a large portion of the time is occupied in doing useful work, that is, in cutting, while a relatively small amount of the time is occupied in returning the tool (waste motion as far as cutting is concerned, but necessary in the operation of any reciprocating cutting tool).

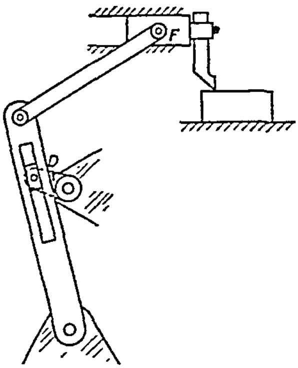


Fig. 57

The possibilities of arranging series of links as mechanisms is limitless. We are here engaged only in a preliminary skirmish to orient our objectives, and to fix some concepts of the appearance and qualifications of the most important elements of mechanisms, and shall leave it to later chapters to probe more intimately the details of their action in applied motion.

22. Change of Position. Relative Motion. Engineering "straight thinking" is characterized by careful attention to its direction. We reason in much the same fashion as we build, establishing first a foundation, then successive levels of carefully selected materials in logical sequence, arriving

finally at a conclusion which may appear as the design of a machine, the plan of a structure, or a crystallization of a basic idea which in turn may be further developed through additional stages of directed thought and active investigation.

The present study of the principles of kinematics should be a planned study; we should establish first a foundation, to contain the elements of definition of basic concepts of motion.

We are accustomed to thinking of the motion of a body as a *change of position.* This definition will become a more satisfactory one when it is

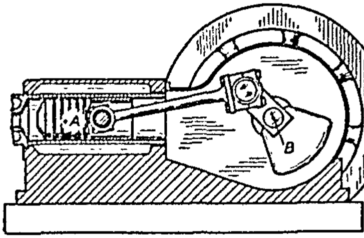


FIG. 58

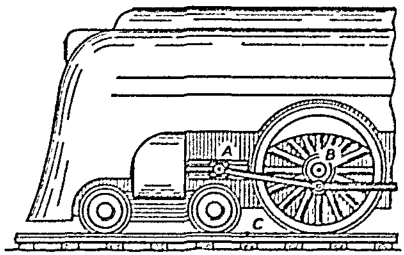


FIG. 59

amplified by furnishing some base or reference to which the position of the body may be definitely related.

It is idle to speak of travelling thirty miles due west, since there is a missing vital element—"from where?"

Adding a base or reference point, for example, the name of a city like Cambridge, gives the previous description of a motion real significance.

The reference point may be, as in the case cited, a location upon the earth's surface.

In the case of a machine, a reference point may be selected upon the frame of the machine and changes of position of moving parts may be described with respect to the selected reference point.

For example, in Fig. 58 we note point A, on a moving piston, and point B at the center of the crankshaft. Point B is stationary, and is a convenient reference point for use in describing the changing position of A. When we announce that A has moved 2 in. to the left *relative to B*, we have completely defined the change or difference between the initial and final positions of A.

All motion may thus be described as relative to a point or other reference which, like B of our illustration, serves to make definite the concept of change of position.

It is frequently necessary to describe a change of position when the reference point is itself in motion. An example of this type is made available by mounting an engine, like that of Fig. 58, upon a locomotive travelling along a track, as in Fig. 59. We now find that a description of the position

of point A in *space* must recognize two contributions to its total amount. As before, A is moving relative to B. In addition, B is moving relative to points on the fixed track, like C.

The total change of position of A *relative to* C is the sum of its motion relative to B and the motion of B relative to C.

The distinguishing name, absolute motion, is given to any change of position which is referred to a point on the earth as reference point. The motion of B relative to C is then an absolute motion, for C is a reference point fixed on the earth's surface. The motion of A relative to B is a relative motion, which is not absolute because the reference point is not fixed to the earth's surface.

If the change of position of A is described by using C as a reference point, instead of first relating it to B and, in turn, relating B to C, the *absolute* motion of A will have been announced.

23. Theorem of Absolute and Relative Motion. The two methods of describing change of position which have been discussed in the preceding article lead to a conclusion, which will be called:

Theorem III. The absolute motion of a moving point is equal to the sum of the motion of the point relative to another moving point plus the absolute motion of the second point.

This theorem may be expressed as an equation:

$$M_A = M_{A/B} \nrightarrow M_B$$

in which M_A is the absolute motion of a point A

$M_{A/B}$ is the motion of A relative to a second moving point B

and M_B is the absolute motion of B.

This conclusion may be restated to serve in a great many kinematic concepts. First, it will apply to bodies as well as to points; in addition, various properties of motion, such as displacement, velocity, and acceleration, may be substituted in the statement for the general term *motion*.

24. Degrees of Freedom. Constraint. In relating motion of a particle to a fixed point, we must recognize the quality of the various possibilities of such motion.

If a complete description of the motion may be given by specifying only one dimension in describing its change of position, the particle has but one *degree of freedom*.

For example, a particle which is free to move only along a straight line

through the reference point has but one degree of freedom, and only one dimension in space—distance along the straight line—describes the motion. A particle forced to move in a circular path about a reference point has also but one degree of freedom, since giving the angle which the radius containing the point makes with a fixed axis through the reference point in the plane of the motion determines the location of the particle.

When a particle is free to move anywhere in a plane, two dimensions are necessary to specify the location, at any instance, of the moving particle, and it has two degrees of freedom.

A particle free to move anywhere in space requires three describing dimensions, and hence has three degrees of freedom.

This concept of "degrees of freedom" is the kinematical counterpart of geometrical descriptions, and it parallels the three dimensions of any space description.

In expanding this thought of "degrees of freedom" to include the rigid body, we observe that a rigid body may have angular motion about an axis, and, at the same time, be free to move in the direction of the axis.

Then, a rigid body may have six degrees of freedom—it may have angular motion relative to each of the three coordinate axes, and motion in the direction of each of the three axes.

In order that motions of particles or of bodies may be obliged to be definite, or determinate, they must be confined by the action of other bodies in contact with them. The influence of these other bodies in causing motion in definite paths is called *constraint.*

A pair of fixed guides serving to force a sliding block to move in a direction parallel to the guiding surfaces is an example of constraint. A pin, about which a wheel may turn, is another example of constraint.

25. Translation. When a body moves so that no straight line in the body changes its inclination, the motion is said to be *pure translation*, or simply translation.

If the body containing line AB (Fig. 60) moves from position *1* to position *2* so that line AB has a constant inclination with the X-axis, the body has moved in translation.

We note that all points lying in AB (or, indeed, anywhere on the body) have moved along parallel paths.

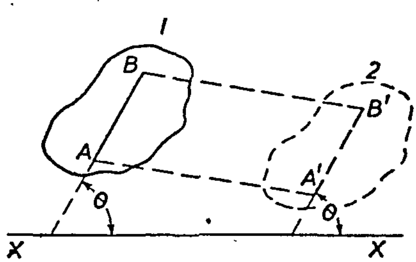


FIG. 60

When these parallel paths are straight lines, as in Fig. 61, the translation is *rectilinear.*

When the parallel paths are curved, as in Fig. 62, the translation is *curvilinear.*

An example of translation as applied in mechanisms is the sliding-block element of the crank-and-connecting-rod mechanism. This is illustrated in Fig. 63. The block S is constrained so that it must move in the fixed guides. Motion is transmitted by the connecting rod BC from the crank AB, which is rotating about a fixed axis A. The block has motion which conforms to the description of rectilinear translation.

Applications of this fundamental mechanism are found wherever it is desired to change from rotation to translation, or vice versa. The piston,

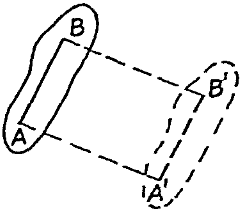


Fig. 61

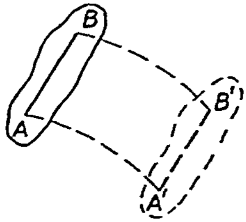


Fig. 62

connecting-rod, and crank of an automotive engine, illustrated in Fig. 56, form such an application.

For an example of curvilinear translation, we may turn to the parallel-

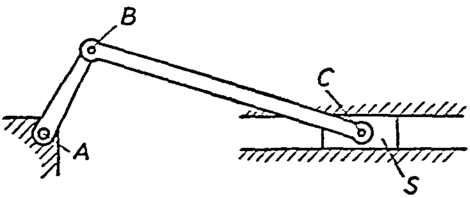


Fig. 63

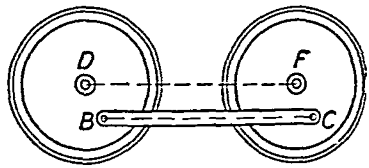


Fig. 64

rod, BC, mounted upon two wheels, D and F, which is illustrated in Fig. 64. BC is parallel to the line connecting the axes of D and F.

In this case, we find that while no straight line (for example, BC) will change its direction, the paths of individual particles are curved. The driving wheels of locomotives are joined by such parallel-rods.

Let us very carefully observe that the story of the motion of one particle of a body in translation becomes a common description, applying with equal truth to all particles.

26. Rotation of a body is defined as motion of the body when one point of the body (or an axis through that point) remains fixed while all other points describe circular paths about the fixed axis.

The plane in which the mass-center of the body rotates is called the *plane of motion,* and the fixed axis is perpendicular to the plane of motion (Fig. 65).

The crank AB of the crank-and-connecting-rod mechanism which was

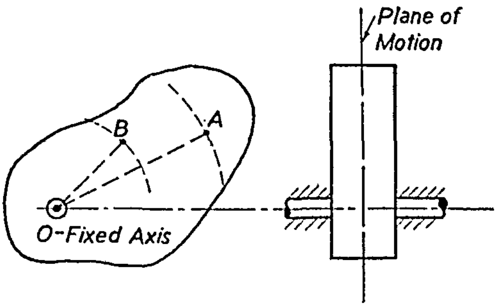


FIG. 65

discussed in Art. 25 is an example of pure rotation, as is every wheel or other body which turns about a fixed axis.

27. Plane Motion. When a body moves so that every particle remains at a fixed distance from a fixed plane, the motion is called *plane motion.*

Figure 66 shows a body moving in plane motion.

The plane in which any point, such as A, moves is parallel to the plane

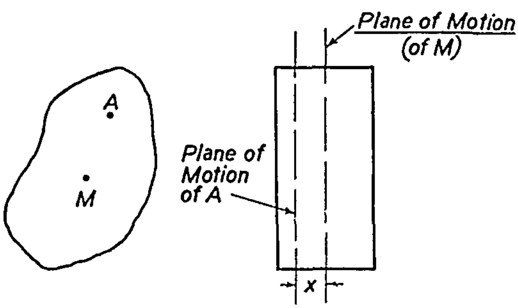


FIG. 66

of motion of the mass-center M, and A remains constantly at distance x away from the plane of motion.

In the most general type of plane motion a straight line in the body changes its inclination, but not about a fixed axis, so that we have a case which is neither pure translation nor pure rotation, but may be considered

a resultant motion to which the change in direction of the line about some axis contributes an element of rotation, while the motion of that axis contributes an element of translation. This very general form of motion will be discussed in the chapters devoted to velocity and acceleration, where more detailed investigation may be made.

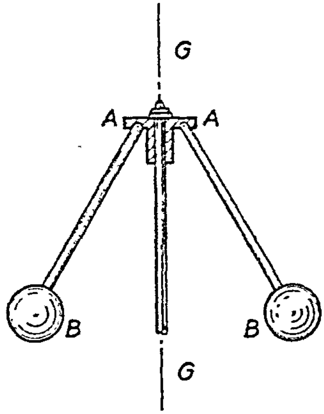


Fig. 67

The connecting-rod, BC, of the crank-and-connecting-rod mechanism of Art. 25 is an example of plane motion.

28. Helical Motion. When a threaded screw turns in a fixed nut, a point on the surface of the thread will move on the surface of a cylinder at the same time that it advances in a direction parallel to the axis of the nut. Such motion is called *helical motion*.

29. Spherical Motion. A steam-engine governor is shown in Fig. 67. As the arms A swing, they raise or lower the particles comprising the ball-weights B relative to the central axis of the governor G–G. The arms themselves are free to rotate about the governor axis, and all points of the balls will therefore be travelling upon surfaces of spheres as the governor is in motion. Such motion of points is *spherical motion*, since a sphere is the surface generated by a point moving at a constant distance from a fixed point.

The cases of helical and spherical motion occur but rarely in mechanism applications. While they may occur in connection with the paths of points, these points will most generally be found to lie upon bodies which themselves have plane motion, as for example, in the case of bevel gears.

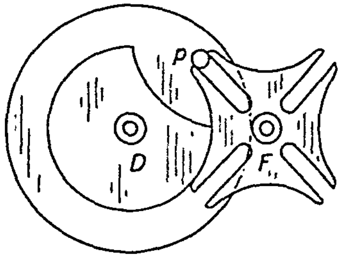


Fig. 68

30. Intermittent Motion. The mechanisms thus far discussed have been in general applications of continuous motion.

In the design of automatic machinery, provision must frequently be made for intermittent motions. These motions are generally synchronized so that some members of the machine will automatically come into action, perform their function, and then remain at rest until a later period when they repeat their performance.

One of the primary bases of intermittent action is the Geneva wheel illustrated in Fig. 68. A driving wheel, D, carries a pin, p, which comes into

contact with a slot of the follower, F, once in every revolution of the driver. The pin and follower remain in contact, in this case, during one quarter-turn of the follower. During the remaining time of one revolution of the driver, the follower remains at rest, locked in position by the circular arcs.

Varying the number of slotted arms on the Geneva wheel follower offers a choice of time element during which the follower will be in motion.

V

DISPLACEMENT

31. Linear Displacement. If a particle moves from point A to point B (Fig. 69), the change of position of the particle is defined as the displacement Δs. This definition confines itself to a description of the *change* of

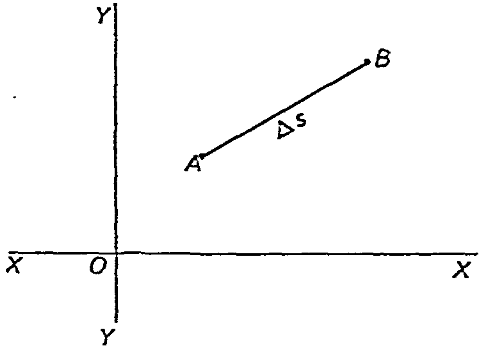


Fig. 69

position. It gives no indication of the exact path over which the particle has travelled. We learn only that the particle started from point A, and

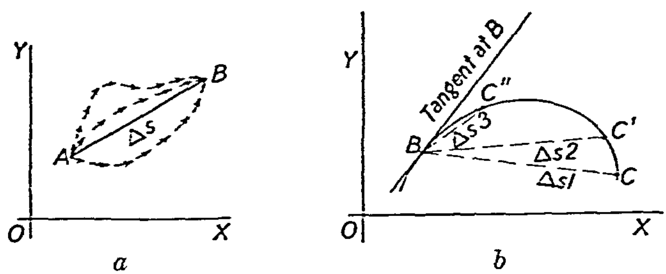


Fig. 70

finally arrived at point B. It might have travelled along any one of an unlimited number of paths, as suggested in Fig. 70-a.

In each of these cases we note that the description of displacement, Δs, summarizes the change of position without describing the intermediate path; and that Δs remains the same for all possible paths of travel.

This displacement is a vector quantity, since it has both magnitude and direction. It should also be noted that the displacement is independent of

the framework of axes employed. While a specific framework is shown in the figure, any other framework might be selected, and would serve equally well in defining the displacement.

The inclination of displacement of a particle which is moving along a curved path may be established by considering the path BC of Fig. 70-b. If the distance BC is decreased to successively smaller intervals BC', BC'', etc., it will be noted that the inclination of Δs, which is always a chord of the curved path, approaches the tangent to the curve as a limit. Then the inclination of displacement of a particle which is moving along a curved path is, at any instant, tangent to the curve.

In most studies of mechanisms, displacements are established graphically by drawing the mechanism to scale in a series of positions.

For example, it is desired to determine the linear displacement of point C (Fig. 71) as the crank AB rotates, for every 15° position of the crank.

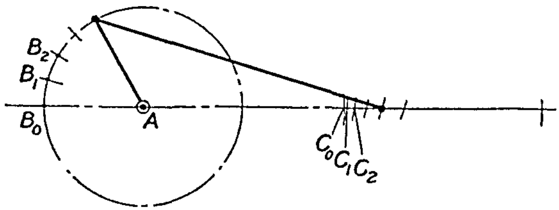


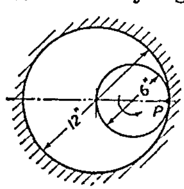
FIG. 71

The path of B is a circle, and points B_0, B_1, B_2, etc., indicate the position of point B at the end of 15° intervals. The length of the connecting rod BC is set as radius, and arcs B_0C_0, B_1C_1, B_2C_2, etc., struck from B_0, B_1, B_2, etc., to intersect the path of point C. These intersections determine the displacements, which are, successively C_0C_1, C_1C_2, etc.

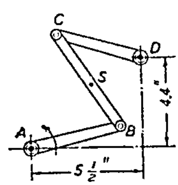
PROBLEMS

67. Plot the path of point P as the inner cylinder rolls without slipping on the inside surface of the annular.

68. The link mechanism shown is a "Watt" straight line linkage. Such a linkage was employed by Watt in his early engines to produce a straight line motion since it was



PROB. 67



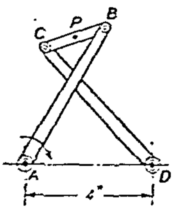
PROB. 68

then difficult to produce accurate plane surfaces in metal, and the crosshead and guide could not be formed with the current tools. AB = 5.0 in.; BC = 5.5 in.; CD = 4.3 in.; BS = 2.54 in.

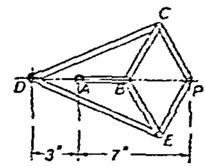
Plot the path of point S from the position when AB is at 30° below its horizontal position until AB is 30° above the horizontal.

69. The Tchebicheff approximate straight line mechanism shown has a driving crank, AB, which rotates about fixed axis A. Plot the path of point P, during the time that AB rotates from a vertical position until CD is in a vertical position. $AB = CD = 10$ in.; $BC = 2$ in.; $CP = 1$ in.

70. The mechanism shown is a Peaucellier's cell linkage. It consists of a series of links pinned to each other. A and D are fixed axes. Plot the path of point P during the time that driving crank AB turns, clockwise, between its limiting positions. $AB = 3$ in.; $BC = CP = PE = EB = 4$ in.



Prob. 69



Prob. 70

32. Angular Displacement. If a line moves so as to change the angle that the moving line makes with a fixed line, this change of position is called *angular displacement.*

In Fig. 72, line OC is shown in an initial position OC and final position OC'. Then the angular displacement $\Delta\theta = \theta_1 - \theta_2$ regardless of the actual travel of the line during the interval. For example, the line OC might first have travelled counter-clockwise from initial position OC to any new position, then clockwise to final position OC'. Its angular displacement remains $\Delta\theta$ which fully describes the completed change of position.

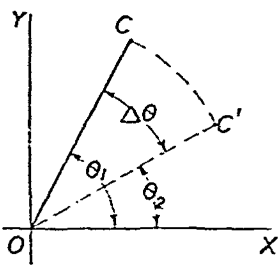
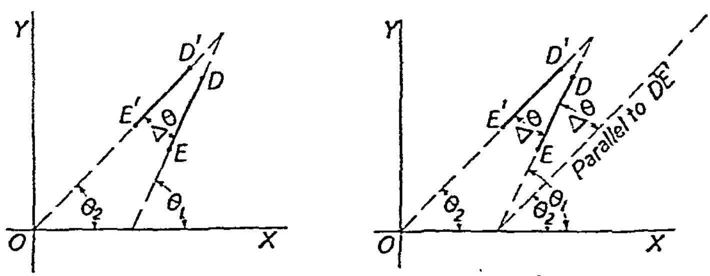


Fig. 72

The illustration of angular displacement which is made in Fig. 72 shows a line rotating about an axis at one end, O, of the line itself.



a b

Fig. 73

The definition of angular displacement is not disturbed if a different origin than the axis of rotation is selected, as in Fig. 73-a.

Here line DE is displaced to position $D'E'$. The angular displacement is $\Delta\theta$, which is again equal to $\theta_1 - \theta_2$ (see Fig. 73-b).

33. Relationship between Linear and Angular Displacements. In this text, the concept of linear displacement will be confined to points, and that of angular displacement will be reserved for lines.

When point C moves to position C' (Fig. 74), the distance which it travels along the arc is CC', subtending angle $\Delta\theta$, and of length $r\Delta\theta$. The linear displacement of the point is Δl.

If now the angle $\Delta\theta$ is indefinitely decreased, it approaches as its limit $d\theta$. The arc $r\Delta\theta$ approaches displacement Δl in length. In the limit, linear displacement $dl = rd\theta$, which establishes the relationship between the angular displacement of a moving line and the linear displacement of the points in the line.

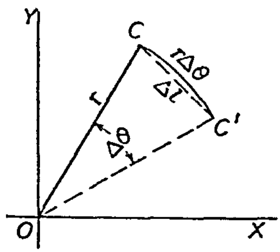


FIG. 74

34. Cams. Geometrical Design for Displacements. Cams have already been described as bodies whose surfaces are shaped so as to transmit desired motions to their followers. We shall first consider the flat plate or disc cam.

The property of displacement of the follower affords a basis for the geometrical design of the cam's surface. Later, in the divisions of our subject devoted to velocity and acceleration, we shall investigate the design of cam surface when special forms of motion are to be obtained for the follower. At present we shall confine our interest to the matter of surface shape, or geometry, of the cam. The geometric design of the surface starts from the desired displacement schedule of the follower as given data. Such data in turn are dependent upon velocity and acceleration, but the geometrical design follows only after the velocity and acceleration properties have been summarized in an announcement of desired displacements, called the *displacement schedule*, and the geometrical application is an independent stage of the design.

Pointed Follower. The simplest form of follower is a pointed rod, guided to travel in a straight line; and the simplest condition of alignment consists of a cam axis lying on the produced straight line path of the pointed follower.

The line joining the cam axis and the starting point of the follower will be called the *reference line*.

The given data consist of the following:
Follower starts at point *1* (Fig. 75).

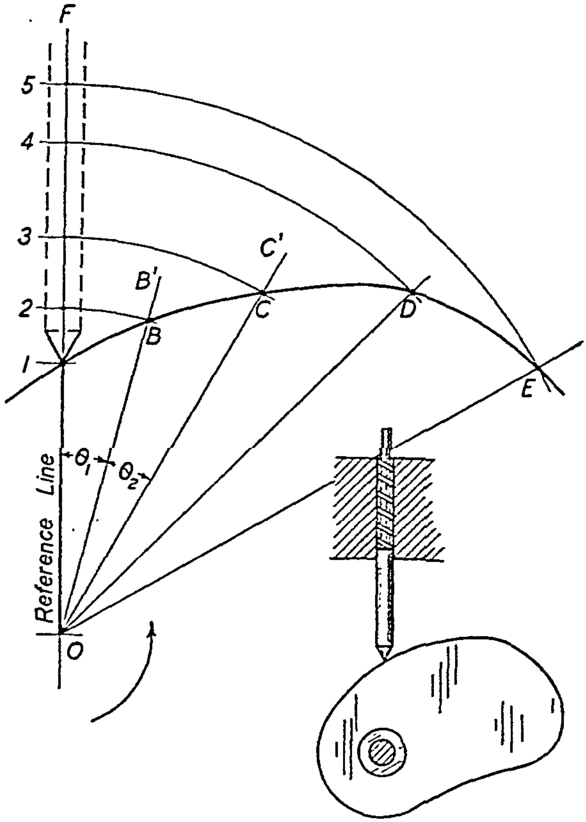


FIG. 75

Cam axis at point *O*, on produced path of follower.
Cam to turn counter-clockwise.

DISPLACEMENT SCHEDULE

Point	*Displacement of tip of follower from point 1*	*Angular displacement of any line on cam from its initial position*
1	0.00 inches	0 degrees
2	0.57 (upward)	15
3	1.30	30
4	2.28	45
5	2.87	6C

This schedule does not, of course, completely describe the displacement schedule for a complete revolution of the cam, but it does carry the design far enough to establish a method of attack.

Having established a straight line OF to serve as the path of the follower, every point, like *1*, *2*, *3*, *4*, *5*, etc., is located to give a complete graphical description of the displacement of the follower, as announced in the displacement schedule.

The next stage, with follower's displacements established, is to consider the cam, whose axis will lie at point O.

At the end of the first announced interval, the point of the follower must be displaced to position *2*, 0.57 inches upward. During this same time interval the cam is turning counter-clockwise, and a line OB' will move through an angle $\theta_1 = 15°$, and lie along the starting or *reference line.*

If now we strike an arc with distance $O2$ as radius, and O as center, this arc will intersect line OB' at point B, a point on the cam surface. Let us check the location of point B to observe its action as one of the points of the surface.

At the start, the point of the follower is at *1*, and the cam is in the position indicated in the figure. During the first time interval the cam will turn counter-clockwise, and the follower's point will remain in contact with its surface, being thus lifted until at the end of the interval line OB will have arrived at the reference position and will have moved the point of the follower to level *2*, since $O2 = OB$. This is the displacement of the follower which was intended and point B is a satisfactory one.

The geometrical design of the cam's entire surface becomes a routine of repetition of the geometry which located point B. For example, if angle θ_2 is made equal to *15°* and an arc struck with radius $O3$ and center O to meet line OC' at point C, we shall have an additional point on the cam's surface. Points D, E, etc., are located by continuing application of the method used for points B and C.

A series of points is thus located. The cam's surface is the locus of these points. As in every case of curve plotting, a sufficient number of points must be taken to insure that the curve represents the true locus within the limits of permissible approximation. The definition of the term "permissible" is governed in the cam example just discussed by the displacement schedule. The design insures proper displacement of the follower at the end of each time interval. Had the displacement of the follower been described at the end of each of smaller time intervals, we should have plotted the number of points on the cam's surface necessary to guarantee a proper performance to correspond with these smaller intervals. The method of design, as far as its geometrical construction aspect is concerned, would not have differed. The most accurate cam surface to accomplish its purpose

With pitch surface established, we now admit the presence of a roller follower by drawing the actual cam surface so that it is always at a distance from the pitch surface equal to the radius of the roller.

The routine of construction is illustrated in Fig. 76. The data given in this design is the same as that for the pointed follower of Fig. 75 plus the

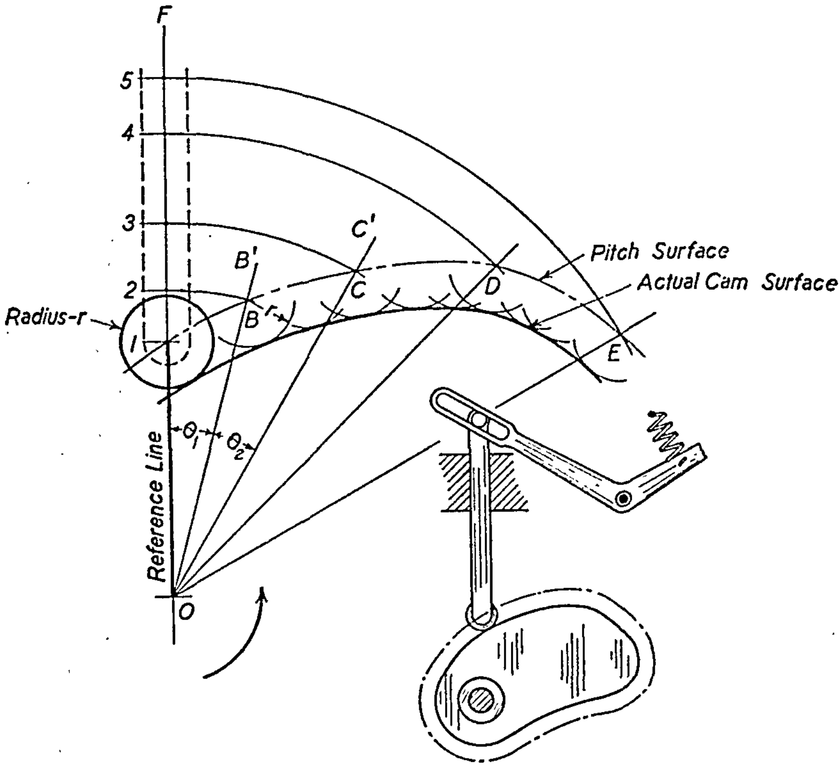


FIG. 76

addition of the radius of a roller with which the follower is to be equipped. This value is given as distance r.

We first repeat the operations of the previous article to obtain the cam surface which would match the displacement schedule of a pointed follower whose point lies at the center of the roller. This cam surface is labelled "pitch surface." From points of the pitch surface, arcs of radius r are now struck.

The curve, or envelope, drawn tangent to the small arcs becomes the *actual cam surface.* Once again, the establishment of an accurate curve de-

pends upon the number of points selected for plotting; in addition, in this case, upon the number of small, radius r arcs which are constructed to serve as guides. In regions of slowly changing curvature, a few arcs will suffice; but where curvature changes rapidly, many will be needed to insure that rapid changes properly influence the final cam surface design.

The size of the cam has, in these examples, been fixed arbitrarily by announcement of the location of the cam's axis. In general, the size is limited only by the details of construction of the machine upon which the cam is to be mounted. Most favorable results are obtained, as far as distribution of thrusts against axes and frictional effects are concerned, by increasing the size of cam relative to size of roller, but such considerations are of importance only beyond the subject of the kinematical design which is of present interest.

The size of the roller must be considered, for its kinematic influence if improperly fixed will be to defeat the success of the intended displacement schedule. The radius of the roller must be less than the smallest radius of curvature on any concave portion of the cam surface, or the roller will bridge over such gaps, and the correct nature of the follower's motion will be disturbed. Figure 77 illustrates such a possibility.

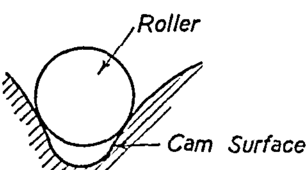


Fig. 77

Flat Followers are widely used, and their properties are readily explored to accomplish the geometrical design of the cam surface. The simplest type is the one shown in Fig. 78. The displacement schedule gives the various levels, *1*, *2*, *3*, *4*, *5*, etc., of the follower. The design is based upon obtaining a series of tangent lines, like A–B, C–D, E–F, G–H, etc., and drawing a curve which shall be tangent to these lines. Let us consider the placing of line A–B. During the first time interval the follower is to rise from its starting position, which is the position shown in the figure, and arrive at level *2*. During this same time interval line OA' will have completed angular displacement θ_1. If, then, an arc is struck with O as center and radius equal to distance O–*2*, this arc will intersect line OA' at point A''. We now erect at A'' a line A–B making the same angle with OA' that the face of the follower makes with O–*1* (in this case 90°). This line AB is the first of a series of tangent lines like C–D, E–F, G–H, etc., which are similarly drawn to match the displacement schedule of the follower at the proper angular displacement of the cam. The cam's surface will be the curve drawn tangent to lines A–B, C–D, E–F, G–H, etc. With this type of follower concave portions of the cam surface cannot be used to accomplish changes of motion since the flat follower would bridge concavities, as in Fig. 79.

When the flat face of the follower in any given position lies at some other angle than 90° with the starting or reference line, the angle which the

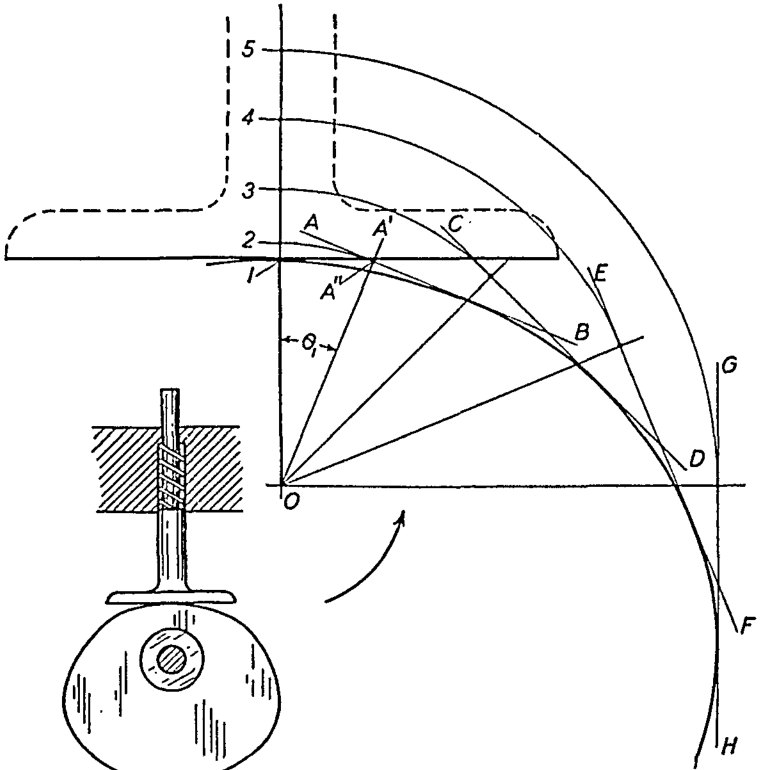


Fig. 78

tangent (like A–B, C–D, etc.) makes with the line which accompanies it to insure correct timing (like OA', OB', etc.) must be the same angle. To clarify this point, Fig. 80 illustrates a cam designed to transmit intended motion to a flat follower whose face is oblique to the path of travel. The angle θ is a fixed angle. Then in locating tangent line AB we proceed by striking an arc from level *2* to intersect OA' at point A''. We next erect line AB so that it makes an angle θ with OA' at A'', and have established the proper tangent. When the cam, rotating counter-clockwise, arrives at

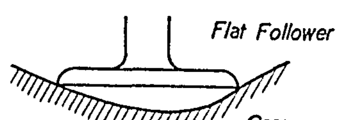


Fig. 79

the reference position its surface will be tangent, as it should, to the face of the follower.

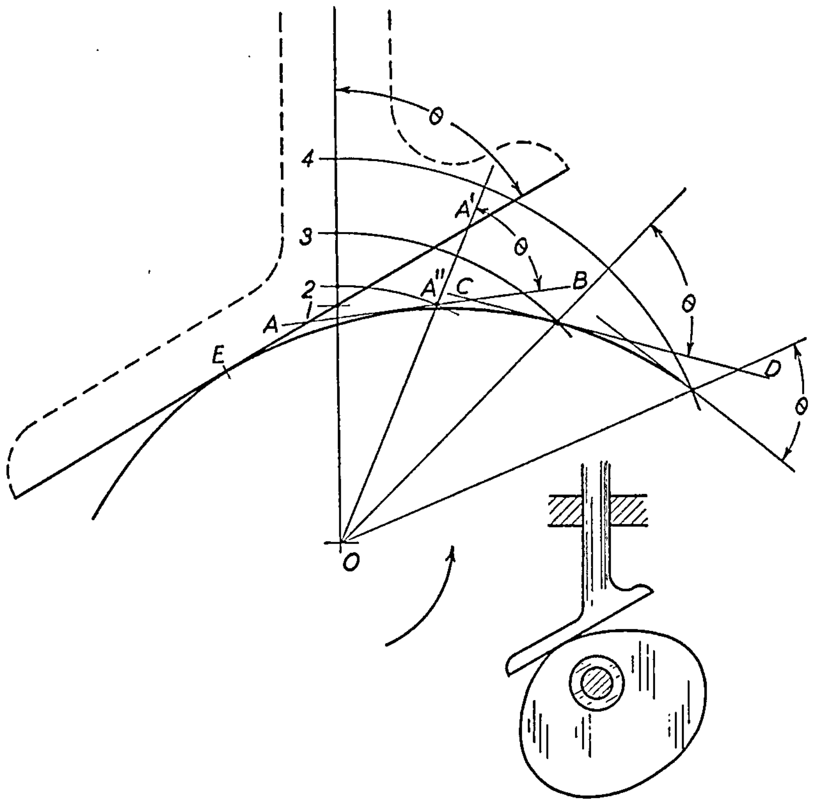


Fig. 80

Or, to further clarify the action, let us assume that the cam remains stationary, and the flat follower is moved from the position shown in solid lines in Fig. 81 to the position shown by dotted lines. Then, when the follower arrives at the dotted position, the cam surface will have advanced the follower to position *3*, which is the intended displacement. The relative motion of the cam and its follower is the same, no matter which of the two units is held still.

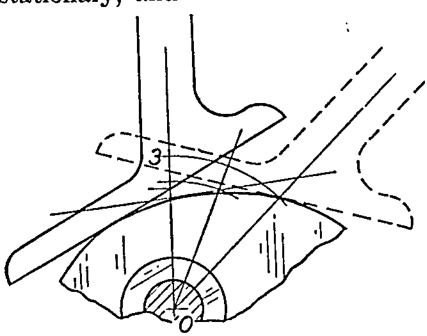


Fig. 81

The contacting flat surface of

the follower should be checked so that it provides sufficient length for contact throughout the complete revolution of the cam. In Fig. 80, for example, all such tangential distances as that between the axis of the follower and the contact point, E, should be measured to determine the maximum follower surface required.

PROBLEMS

Problems 71–76. The displacement schedule for a plate cam is given. The path of the follower is a straight vertical line, and the location of the cam axis, its sense of rotation, and the type of follower is given. Design the plate cam. *Note: The displacement is measured from the starting point, 1, and is labelled (+) for an upward displacement, and (−) for a downward displacement from the starting point.*

71. Follower—Pointed
Cam axis—1 in. below starting position of tip of follower.
Cam rotates clockwise.

DISPLACEMENT SCHEDULE

Note: All displacements of follower are above starting position.

Cam displacement	*Follower displacement*	
0 degrees	0 inches	
15	0.07	
30	0.27	
45	0.69	
60	1.00	
75	1.48	
90	2.00	
105	2.53	
120	3.05	
135	3.58	
150	4.10	Dwell
165	4.10	
180	4.10	
195	4.00	
210	3.70	
225	3.20	
240	2.50	
255	2.50	Sudden drop
255	1.50	
270	1.50	Dwell
285	1.50	
300	1.40	
315	1.10	
330	0.60	
345	0.20	
360	0	

72. Follower—Pointed
Cam axis—2 in. below starting position of tip of follower.
Cam rotates counter-clockwise.

DISPLACEMENT SCHEDULE

Note: All displacements of follower are above starting position.

Cam displacement	*Follower displacement*	
0 degrees	0 inches	
15	0.04	
30	0.22	
45	0.54	
60	0.98	
75	1.54	
90	2.14	
105	2.64	
120	2.92	
135	3.00	
150	3.20	
165	3.96	
180	5.24	
187.5	5.78	
195	6.00	
200	5.86	
205	5.44	
210	5.00	
225	4.60	Dwell
240	4.60	Dwell
255	4.60	Dwell; Sudden drop
255	4.00	Sudden drop
270	3.60	
285	3.24	
300	2.86	
315	2.30	
330	1.54	
345	0.72	
352.5	0.32	
360	0	

73. Follower—Roller, $\frac{3}{4}$ in. dia.
Cam axis—5 in. below starting position of center of follower.
Cam rotates counter-clockwise.

DISPLACEMENT SCHEDULE

Cam displacement	*Follower displacement*
0 degrees	0 inches
15	+0.14
30	+0.42
45	+0.86
60	+1.32
75	+1.70
90	+1.90
105	+2.16
112.5	+2.48
120	+3.00
135	+2.18
150	+1.30
165	+0.52
180	0

Cam displacement	*Follower displacement*
195 degrees	−0.32 inches
210	−0.56
217.5	−0.66
225	−0.80
232.5	−0.60
240	−0.36
255	+0.22
270	+0.88
285	+1.50
300	+1.92
315	+2.00
330	+1.26
345	+0.40
360	0

74. Follower—Roller, 1 in. dia.
Cam axis—4.5 in. below starting position of center of follower.
Cam rotates clockwise.

DISPLACEMENT SCHEDULE

Cam displacement	*Follower displacement*	
0 degrees	0 inches	
15	+0.10	
30	+0.40	
45	+0.90	
60	+1.60	Dwell
75	+1.60	
75	−1.80	Sudden drop
90	−1.80	Dwell
105	−1.80	
120	−1.80	
135	−1.80	
150	−1.80	
165	−2.00	
180	−2.20	
195	−2.10	
210	−2.00	
225	−1.90	
240	−1.80	
255	−1.60	
270	−1.20	
285	−0.80	
300	−0.40	
315	−0.10	
330	0	Dwell
345	0	
360	0	

75. Follower—Flat; face at 90°.
Cam axis—3 in. below starting position of face of follower.
Cam rotates clockwise.

Cam displacement	*Follower displacement*
255 degrees	4.75 inches
270	4.42
285	3.98
300	3.22
315	2.06
330	0.88
345	0.18
360	0

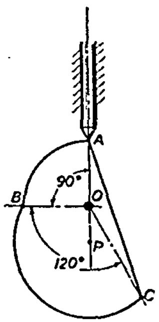
PROBLEMS

Problems. 77–79. The plate cam surface shown in the figure is composed of circular arcs and straight lines. Plot the displacement-time curve of the follower.

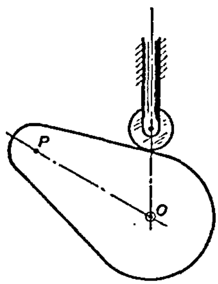
77. Cam rotates, counter-clockwise, about point O, at 200 r.p.m. Radius of arc AB = 4.4 in., with center at O. Arc BC has its center at P. OP = 2.4 in.

78. Cam rotates, clockwise, about point O, at 600 r.p.m. Radius of small arc = 1.2 in., with center at P. Radius of large arc = 3.0 in., with center at O. OP = 6 in. Straight lines tangent to both arcs. Radius of roller follower = 1.0 in.

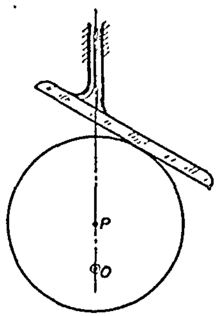
79. Cam rotates counter-clockwise about O, at 1 radian per sec. Diameter of cam = 7.0 in., center at P. OP = 2.2 in. The face of the follower makes an angle of 60° with the vertical stem.



PROB. 77



PROB. 78



PROB. 79

35. Cams. Effect of Offset Axis on Geometrical Design. The examples which have been discussed have been simplified by locating the cam axis directly on the line which is the follower's path.

When the axis of the cam is offset from the straight-line path of the follower, or when the path of the follower is not a straight line, the method of attack must be modified to meet the new conditions.

Here, again, a concrete example will furnish the most effective illustration of method. We may once again use as basis the data of the pointed-follower cam given in Art. 34, with one change. The cam axis, O, is now to be located as shown in Fig. 82, so that it no longer lies on the extended path 1–F of the follower.

The reference line which orients the starting position is drawn from

center O to point *1*, the starting position of the point of the follower. We note that, since the reference line departs from the path of the follower as it moves outward from the cam axis, the method of design used previously will require modification. As the follower point is displaced from *1* to *2*,

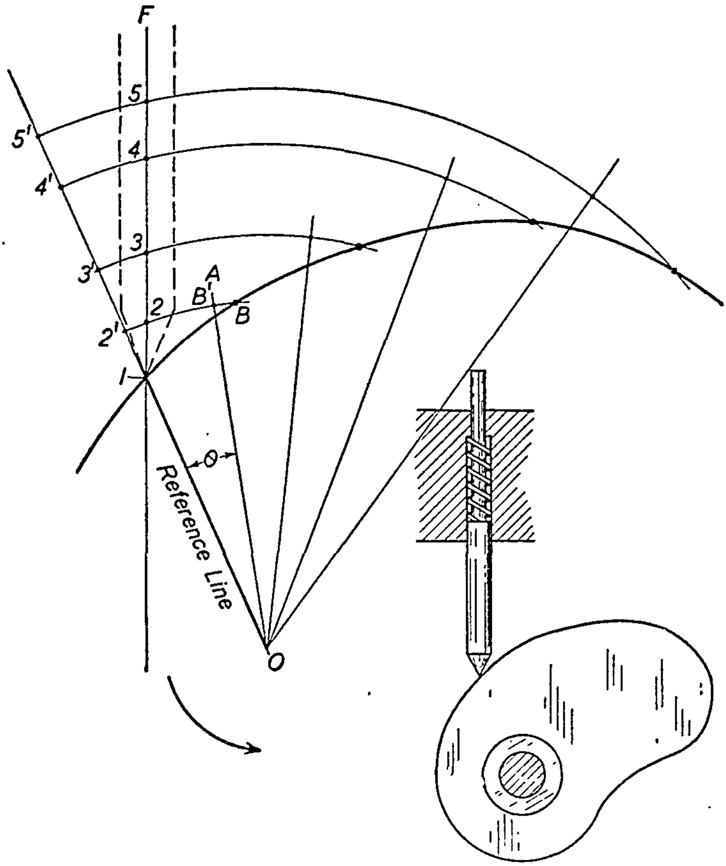


Fig. 82

line OA (with angle θ matching in time the interval for this displacement) will be the line marking proper angular displacement.

We should, therefore, strike an arc of radius $O2$ and center at cam axis, to intersect both the reference line (at point $2'$) and the line OA (at point B') and carry this arc beyond point B' for a short distance. If now we measure distance 2–$2'$ with the dividers, and from point B' set off the same distance, we shall have located point B, which is the first point on the cam's surface. Measuring 2–$2'$ with dividers, we shall have measured the chordal distance, and chord B–B' equal to it will have made arc 2–$2'$ equal to arc B–B' since

both arcs lie on the same circumference. It is of interest to strengthen the understanding of such cam action by observing the motion. Let us imagine a start of the cam at the position drawn in the figure. During the first time interval line OA will have moved to position O–1 and point B will have trailed behind it, remaining always at a distance equal to arc B–B' behind it. When point B' arrives at point $2'$ as it will do at the end of the

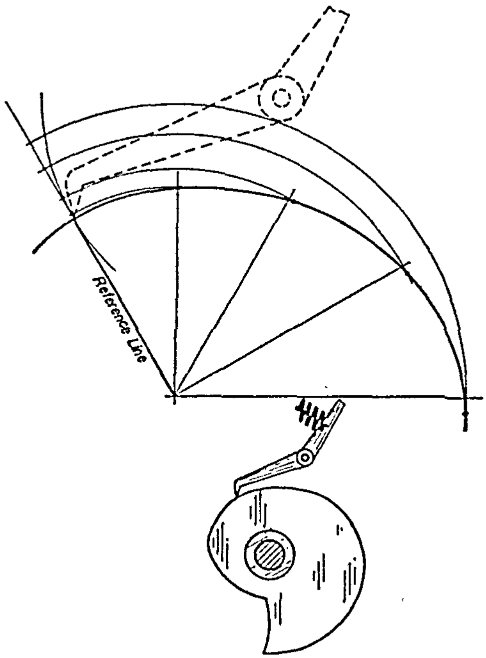


Fig. 83

first time interval, point B will be at level 2. Therefore the cam's surface has been properly designed to move the point of the follower to level 2 in the required time. The continuation of the design, as in all of the preceding cases, involves repeating the construction, each time making allowance for the offset distances, like 2–$2'$, 3–$3'$, 4–$4'$, etc., since the point of the follower follows the vertical path, while the several angular displacement lines are arriving, in every case, at the reference line.

The roller follower or flat follower may also be designed to run with cam axis offset from their path, in which case the allowance for offset is made in exactly the same fashion as in the case of the pointed follower.

Another instance of the use of this offset method arises when the follower is a rocker arm pivoted about a fixed axis, as in Fig. 83. It will be noted

that the path of the point of the follower is now a circular arc with center at the axis of the arm. The reference line here, too, deviates from the follower's path and offset allowances must be made by the method employed in the previous example.

36. Positive Motion Cams. The illustrations of cam motion have thus far included followers which must depend for a completion of their cycle upon a spring, gravity, or some other auxiliary source to maintain continuous contact throughout the cycle. It is possible to provide positive motion through design of the cam and follower.

The principle is illustrated in Fig. 84. A groove, designed with its center line as a pitch surface by the methods previously employed, is cut in the

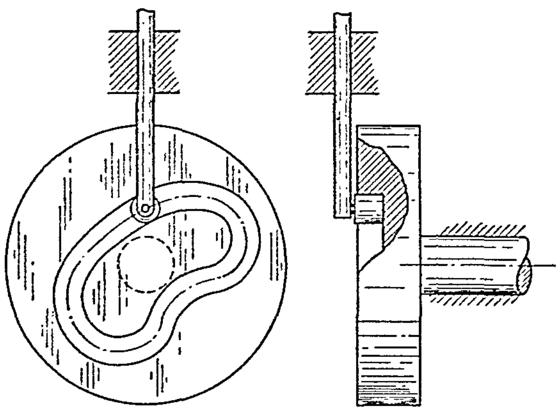


FIG. 84

plate. The follower roller is held in position in the groove, and the motion throughout the cycle of operations is positively established by the cam, without recourse to auxiliary or assisting devices to maintain contact. After designing the pitch surface, arcs of radius equal to the roller radius (or slightly greater for clearance) are employed to establish tangent curves or envelopes which form the sides of the groove.

Another method of securing positive motion is to provide the type of followers shown in Figs. 85 and 86, where one or two cams mounted on the same axis are designed so as to provide the required motion on both up and down strokes to a reciprocating follower.

The cam arrangement of Fig. 86 illustrates a simple manner of providing a positive motion of the follower during the entire revolution of the cam. It consists of two cams—a main and a return cam—both keyed to the same cam shaft.

The main cam has been designed to provide the lifting drive to roller

axis A for a given displacement schedule during one half-revolution. The procedure of geometrical design is identical with that discussed in Art. 34 for any plate cam with roller follower. The return cam is next designed to drive roller axis B for the remaining half-revolution, returning the follower to its starting position to satisfy the balance of the displacement schedule.

To complete the shape of the main cam, the distance AB, which is the fixed center distance of the two roller axes, is laid off along diameters

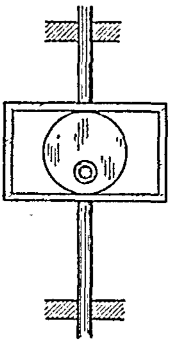


Fig. 85

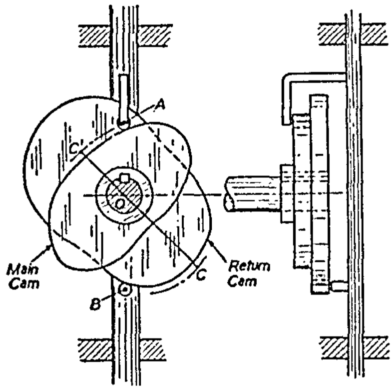


Fig. 86

from the points on the pitch surface of the return cam. The determination of one such point on the main cam is shown. C is any point on the previously determined half-surface of the return cam. On a line from C through O, the axis of the cam shaft, the distance CC' is laid off, establishing point C', one point on the pitch surface of the main cam.

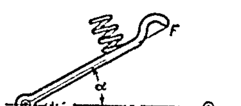
The remainder of the pitch surface of the return cam is established by laying off distance AB from points on the original main cam.

This procedure fixes the shapes of the two cams so that they are constantly in contact with their respective roller followers.

PROBLEMS

80. Design a plate cam for a pointed follower whose path is a straight vertical line with cam axis 2 in. below, and 1.3 in. to the right of the starting position of the follower. Cam rotates counter-clockwise. Use the displacement schedule given in Problem 71.

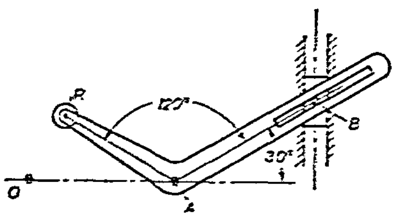
81. Design a plate cam with axis at O, to give the arm carrying the pointer, F, the displacements given in the table. The arm rotates about a fixed axis at A. $AF = 7.0$ in. O is 9.0 in. to the right of A. Cam rotates clockwise.



Prob. 81

DISPLACEMENT SCHEDULE

Cam displacement	*Follower displacement (angle α)*	
0 degrees	0 degrees	
15	5	
30	15	
45	30	
60	45	
75	60	
90	70	
105	73	
120	70	
135	65	Dwell
150	65	
165	65	
180	65	
180	42	Sudden drop
195	40	
210	34	
225	24	
240	14	
255	8	
270	2	
285	0	Dwell
300	0	
315	0	
330	0	
345	0	
360	0	



PROB. 82

82. Design a positive-motion cam for the roller follower shown. The roller is pinned to a rocker arm which rotates about a fixed axis, A. The block at B slides in fixed guides, and carries a pin which slides in the slot of the rocker arm. Cam turns clockwise about axis O. The cam is to be of the positive-motion type shown in Fig. 84. $RA = 6.0$ in.; $AO = 7.0$ in.; vertical path of B is 6.0 in. to the right of A. Roller dia. $= \frac{3}{4}$ in.

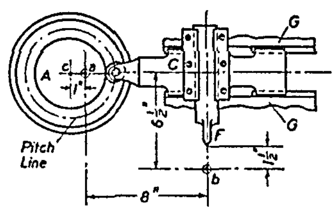
DISPLACEMENT SCHEDULE FOR B

Cam displacement	*Follower displacement (from position shown in figure)*
0 degrees	0 inches
15	0.1
30	0.4
45	0.9
60	1.6
75	2.3
90	2.8
105	3.1

Cam displacement	*Follower displacement (from position shown in figure)*	
120 degrees	3.0	inches
135	3.0	Dwell
150	3.0	
165	3.0	
180	2.9	
195	2.6	
210	2.1	
225	1.4	
240	1.3	
255	1.2	
270	1.1	
285	1.0	
300	0.9	
315	0.8	
330	0.5	
345	0.2	
360	0	

83. Cam A is a positive motion plate cam. It rotates clockwise at 5 r.p.m. about axis a, driving carriage C in a horizontal reciprocating motion on fixed guides G. The pitch line of A is a circle of 6 in. dia. with center at c. Cam B is to be a plate cam. It rotates counter-clockwise at 5 r.p.m. about axis b, giving a vertical motion to pointed follower F, which slides in guides attached to C.

Plot the path of point F, and design cam B. F is shown at its starting position.



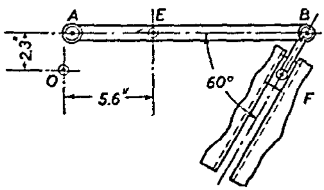
PROB. 83

VERTICAL DISPLACEMENT SCHEDULE FOR F

Displacement of cam B	*Vertical displacement of F (from starting position shown)*	
0 degrees	0	inches
15	0.05	
30	0.19	
45	0.42	
60	0.75	
75	1.13	
90	1.50	
105	1.88	
120	2.25	
135	2.58	
150	2.81	
165	2.95	
180	3.00	
195	3.00	
210	3.00	
225	3.00	Dwell
240	3.00	
255	3.00	
270	3.00	

Displacement of cam B	*Vertical displacement of F (from starting position shown)*
285 degrees	2.80 inches
300	2.25
315	1.50
330	0.75
345	0.20
360	0

84. Initial Cam. Draw the letter "*A*" 5 in. in height. Design a pair of matched plate cams to cause a pointed follower to trace the letter. One cam is to supply the horizontal component motion; the other is to supply the vertical component motion simultaneously.



PROB. 85

85. Design a plate cam, with axis at O, to rotate counter-clockwise. The arm AB rotates about a fixed axis at E, and drives the sliding block at F through link BF. F slides in fixed guides. $AE = 5.0$ in.; $EB = 9.5$ in.; $BF = 3.0$ in. Diameter of roller at $A = 1.25$ in.

DISPLACEMENT SCHEDULE FOR *F*

(Displacements along fixed guides)

Cam displacement	*Follower displacement (from starting position shown)*
0 degrees	0 inches
15	0.10
30	0.38
45	0.84
60	1.50
75	2.25
90	3.00
105	3.75
120	4.50
135	5.16
150	5.90
165	6.00 } Dwell
180	6.00 } Dwell
195	5.90
210	5.60
225	5.13
240	4.50
255	3.78
270	3.00
285	2.22
300	1.50
315	0.87
330	0.40
345	0.10
360	0

86. A main and return cam of the type shown in Fig. 86 is to rotate about axis O. The vertical axis of the yoke follower is 1 in. to the left of O. The distance between centers of the roller followers is 8 in. and the rollers have 1 in. dia. The cam rotates counterclockwise and the starting position of the center of the upper roller is 2 in. above O.

UPWARD DISPLACEMENT SCHEDULE

Cam displacement	*Follower displacement*	
0 degrees	0 inches	
15	0.2	
30	0.8	
45	1.8	
60	3.2	
75	3.6	
90	4.0	
105	4.4	
120	4.8	
135	5.2	
150	5.6	Dwell
165	5.6	
180	5.6	

DOWNWARD DISPLACEMENT SCHEDULE

Cam displacement	*Follower displacement*	
195 degrees	5.5 inches	
210	5.1	
225	4.6	
240	3.9	
255	3.0	
270	2.0	
285	1.0	
300	0.5	
315	0.3	
330	0.1	
345	0	Dwell
360	0	

37. Cylindrical Cams are employed when the follower is to move parallel to the axis of the cam itself. The design of cylindrical cams, like that of plate cams, consists of two stages. First, the desired motion of the follower to provide displacement conforming to announced velocity and acceleration patterns is investigated. to establish a displacement schedule. When the displacement schedule is available, the design becomes an application of geometry to form the cam surface. In

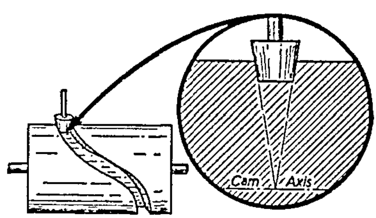


FIG. 87

this case, the motion of the follower is determined by the shape of a

groove cut in the surface of the cylinder. The follower is usually equipped with a conical roller whose apex is at the axis of the cylinder. This provides rolling contact (Fig. 87).

The radial sides of the groove are fashioned with a cutting tool shaped to conform to the desired dimensions of follower roller, and the geometrical design need fix only the position of a center line of the groove which will in turn fix the path of travel of the cutting tool. This is quite equivalent to the method of attack which was employed when we analyzed the procedure of design for a roller follower and plate cam, where a pointed follower was substituted to orient a pitch cam surface.

With an announced displacement schedule for a pointed follower, the groove center line for a cylindrical cam becomes the objective of the design, leaving it to the manufacturing process employing a properly shaped grooving tool to complete the correct fashioning of the sides of the groove.

Given the following displacement information:

A follower moves in translation and has reciprocating motion; while the cam has a continuous motion of rotation about a fixed axis at constant angular velocity.

The follower is to move to the right, remain or "dwell" there for a given period of time, and then return to its starting position, while the cam makes one complete revolution.

Displacement Schedule. The starting position is that shown in Figs. 88 and 89, where the follower is at position *O* while the diameter *YY* (see side elevation) is vertical. The cam is to rotate counter-clockwise.

Position *14* is identical with position *O*, and the follower has returned to the starting position. Continuation of the motion is a repetition of the schedule for the first cycle of events.

Follower moves to position	*While cam turns*
0	0 degrees
1	15
2	15
3	30
4	30
5	20
6	20
7	10
8	10
8 Dwell	30
9	30
10	20
11	20
11 Dwell	20
12	30
13	30
14 (*O*)	30

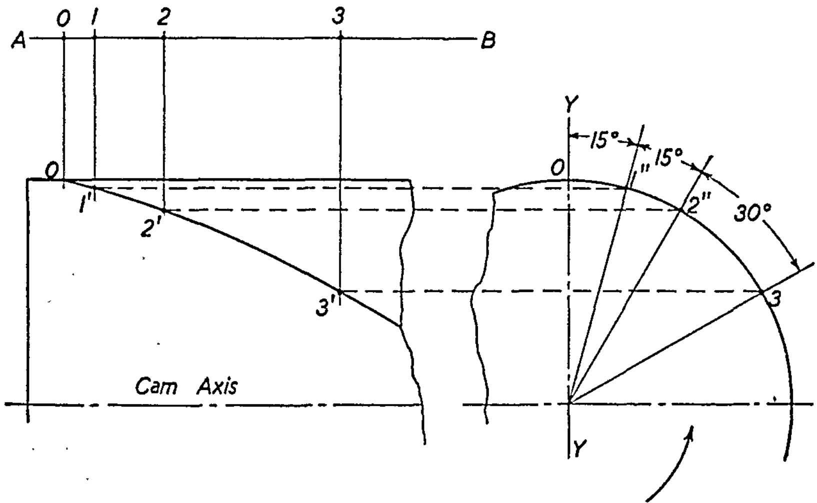


FIG. 88

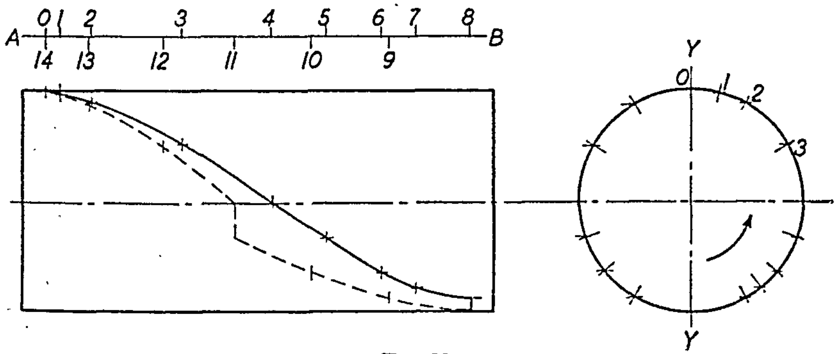


FIG. 89

Any line, like AB, parallel to the axis of the cam, is first divided into the displacements given in the schedule. For example, displacement *0–1* is laid off along this line, and a projector, perpendicular to the cam axis, is dropped to the front elevation of the cam. An angle of 15° is laid off clockwise from line YY, and a projector from *1''*, parallel to the axis of the cam, is drawn to meet the vertical projector from *1* at point *1'*, which is the first point on the center line of the groove. The next displacement is to position *2* which is plotted on AB, and a corresponding angle of 15° is laid off from the preceding position on the side view. Projectors from these two positions—linear and angular—locate point *2'*, the second point on the center line of the groove. We continue through the displacement schedule, matching linear displacement of follower against corresponding angular

displacement of cam. The locus of the points which like *1'* and *2'* lie on the cam surface is the required center line of the groove. The completed design is shown in Fig. 89. It should be noted that the dwell at point *8* is provided by a line perpendicular to the axis of the cam; such a path allows

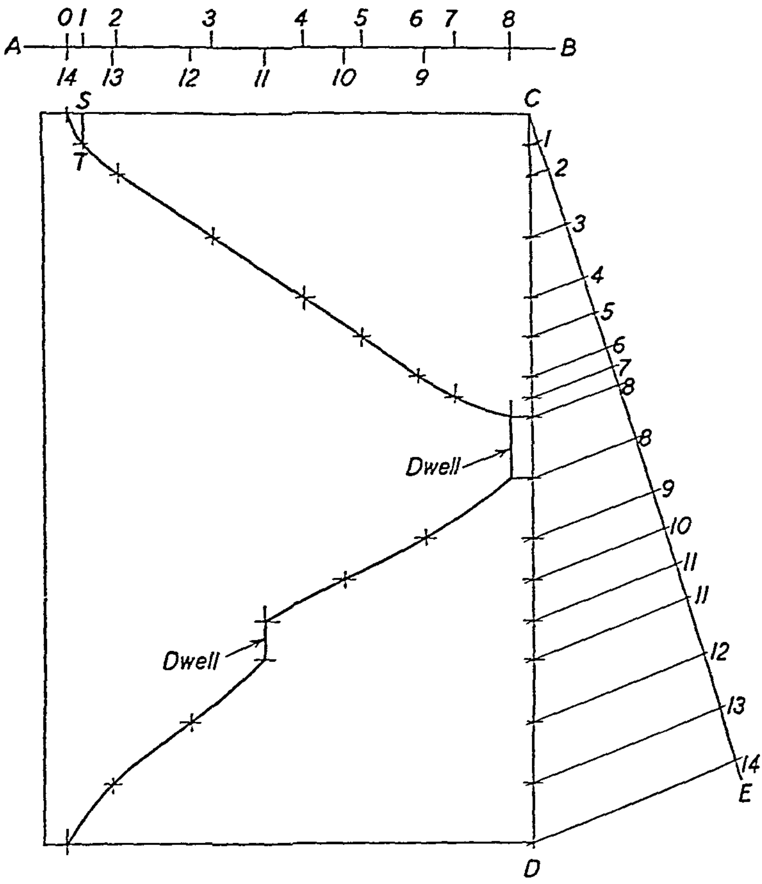


Fig. 90

the cam to rotate while the follower remains at position *8*. Another "dwell" occurs at position *11*.

In manufacturing the cam itself, a template giving the developed surface is useful. The development of the surface of the cam just designed is shown in Fig. 90.

This device gives the true lengths of "spotting" or locating distances on the cam surface, while the previous design in Fig. 89 showed the projected elevation.

The development of the surface is an application of the principles of descriptive geometry. The location of point *1′* on the development will indicate the method. If a projector from *1′* of the front elevation (see Fig. 88) is carried down to the development, and line *ST* made equal to the true length of circular arc *O–1″* of Fig. 88, which is obtainable from the side elevation, we shall have established the coordinates of point *T*, on the development.

The length of ST = arc *O1″* of the side elevation may be taken by rectifying the arc into the straight line by subdividing the arc with the dividers into a large number of small portions which are then transferred. This is an approximation, since the sum of chordal distances does not equal the true length of arc. In the interests of greater accuracy, the method shown in Fig. 90 is an improvement.

The total length of cam circumference is computed, and drawn as line *CD*. At any angle with *CD*, a line *CE* is drawn, and on *CE* units are laid off (any convenient unit of a scale is satisfactory) in the same ratio as the angular displacement intervals of the displacement schedule. If the last point, *14*, is then joined with point *D*, and parallels to line *14–D* drawn from each division, *13*, *12*, *11*, etc., the line *CD* will be divided into segments which are equal to the lengths of arcs *O1″*, *1″–2″*, etc.

When the developed pattern is wrapped around the cylinder of the cam, it will fix the position of the center line of the groove.

PROBLEMS

87. A cylindrical cam has a diameter of 5 in. Design the cam for the following displacement schedule, in which values of the displacement of the follower for each 15° angular displacement interval of the cam are given.

The stroke of the follower is a straight line parallel to the cam axis, 8 in. long.

Cam displacement	*Follower displacement*	
0 degrees	0 inches	
15	0.16	
30	0.60	
45	1.24	
60	2.00	
75	2.77	
90	3.40	
105	3.84	
120	4.00	
135	4.20	
150	4.80	
165	5.50	
180	6.20	
195	7.20	
210	7.80	
225	8.00	Dwell
240	8.00	Dwell

Cam displacement	*Follower displacement*
255 degrees	7.83 inches
270	7.48
285	5.74
300	4.00
315	2.26
330	0.60
345	0.16
360	0

88. Design a cylindrical cam, 4 in. diameter. The stroke of the follower is a straight line parallel to the cam axis, 6.5 in. long.

Cam displacement	*Follower displacement*	
0 degrees	0 inches	
15	0.25	
30	1.00	
45	2.25	
60	4.25	
75	5.50	
90	6.25	
105	6.50	Dwell
120	6.50	
135	6.50	
150	6.50	
165	6.40	
180	6.10	
195	5.60	
210	5.10	
225	4.80	
240	4.70	Dwell
255	4.70	
270	4.70	
285	4.30	
300	3.40	
315	2.80	
330	2.10	
345	0.10	
360	0	

38. Screw Threads. Differential Screws. The evolution of the inclined plane into the screw thread has been described in Art. 20.

While the major use of threaded screws is found in fastening devices, they are also used as mechanisms for converting angular displacement into linear displacement.

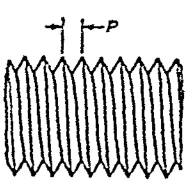


Fig. 91

The *pitch* of a screw thread is the distance p of Fig. 91, which is the distance from the top of one thread to the top of the adjacent thread.

Threads may be of the single-thread or multiple-thread forms illustrated in Fig. 92. In all cases the pitch is the distance between tops of adjacent threads.

The *lead* is the linear displacement or distance the threaded screw will advance into a tapped hole for one revolution of the screw.

Right- and left-handed screw threads are shown in Fig. 93. The distinctive appearance of each will be noted there. The helices of a right-handed screw slope upward to the right as one looks at the near side of the screw; those of the left-handed thread slope upward to the left. The kinematic distinction between the two types arises from the directions of the helices. If a right-handed screw is inserted in a similarly tapped hole it will advance

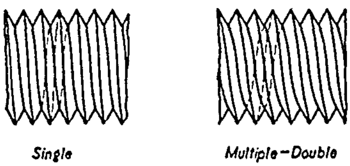


FIG. 92

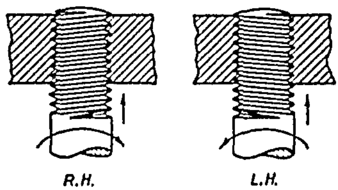


FIG. 93

into the hole when turned clockwise. The left-handed screw must be rotated counter-clockwise to advance into a similarly tapped hole.

The *differential* screw consists of a single piece upon which threads of different lead have been cut, as in Fig. 94. The kinematic influence of the differential screw may be noted by considering the effect upon the distance

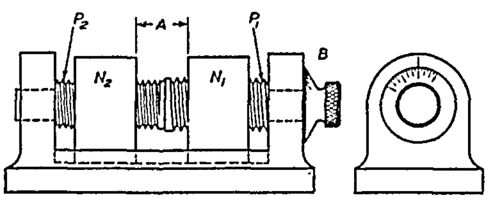


FIG. 94

A of rotation of the screw. The two nuts N_1 and N_2 are constrained by the guides so that their motion must be one of translation only.

The lead of thread P_1 is 0.06 inches, R.H., and that of P_2 is 0.05 inches R.H. If the screw is given one revolution clockwise, as viewed from end B, the nut N_1 moves 0.06 inches to the right, while N_2 moves 0.05 inches to the right. For every revolution of the screw, then, the distance A is increasing 0.01 inches. If a dial indicator is mounted so as to record fractional parts of a revolution of the screw, a reading of 1 degree will indicate a change in distance A of

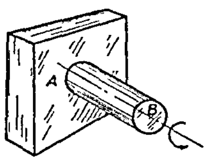
$$\frac{1}{360} \times \frac{1}{100} = \frac{1}{36{,}000} \text{ inch.}$$

The principle which has been applied indicates the possibility of employing differential screws to accomplish or to measure linear displacements with a high degree of sensitivity.

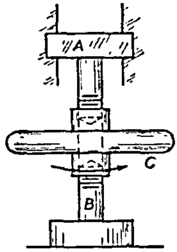
PROBLEMS

89. The block A is stationary. If the threaded piece B is inserted in a tapped hole and rotated in the direction shown it will advance into piece A a distance of 1 in. in 4 revs. of B. The thread is a double-thread. Give its lead, pitch, and direction (L.H. or R.H.).

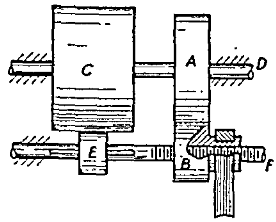
Ans. $\frac{1}{4}$ in.; $\frac{1}{8}$ in.; L. H.



Prob. 89



Prob. 91



Prob. 93

90. If B of Problem 89 is triple-threaded, and has a pitch of 16 threads per in., how far will it move, axially, in 4.3 revs.?

91. A has a lead of 0.3 in., L.H. Find the lead of the screw on B if 100 turns of wheel C in the direction shown raise A 2.0 in. B is stationary, and A is guided by fixed guides.

Ans. 0.28 in. L. H.

92. In the differential screw mechanism of Problem 91, 200 turns of wheel C in the direction shown are required to raise A 1.0 in. A has 10 threads per in. L.H. Find the pitch of the single-threaded screw on B. What pitch must B have to lower A 1.0 in. in 300 turns of C?

93. The feed mechanism is used on a drilling machine to feed the drill into the work as it rotates. D is the drive shaft, and gears A and C are fastened to it. F is the follower shaft which carries the drill chuck. Gear E is keyed to F so that it must rotate with F but can slide axially along it. Shaft F has a screw thread of $\frac{1}{8}$ in. lead R.H. inserted in a tapped hole in gear B. B is attached to the frame so that it may rotate relative to F but cannot move left or right. E makes 3 revs., and B 2 revs., while D makes 1 rev.

How far will F advance axially when D makes one rev.?

94. Redesign the screw thread of Problem 93 so that its lead will cause F to have a feed of $\frac{3}{32}$ in. per rev. of D.

39. Absolute and Relative Displacements. The concept of absolute and relative motion discussed in Art. 23 and summarized as Theorem III may be applied in a study of displacements.

To illustrate relationships, let us first consider the sliding-block mechanism shown in Fig. 95.

A large block T is mounted between fixed guides, which constrain the block so that all particles can move only in the direction of axis ZZ.

Since the guides are fixed, any motion which is described as "relative to the guides" is absolute motion.

A slot is cut in block T and a small block carrying point A inserted so that it may slide in this slot.

The starting position of the block T is shown in solid outline. Point A is in contact with point D of the large block T.

If we now move the small block (while holding T at rest) so that point A goes to position A', the displacement of A relative to D is AA'. This is an absolute displacement, for block T and its contained point D has no motion relative to the guides.

We next set the block T in motion, so that D is moved to D', allowing A to remain still relative to D. A will be displaced from A' to A''. The displacement $A'A''$, which is equal to DD', is an absolute displacement for this is displacement relative to the fixed guides. The new position of both blocks is shown in dotted outline.

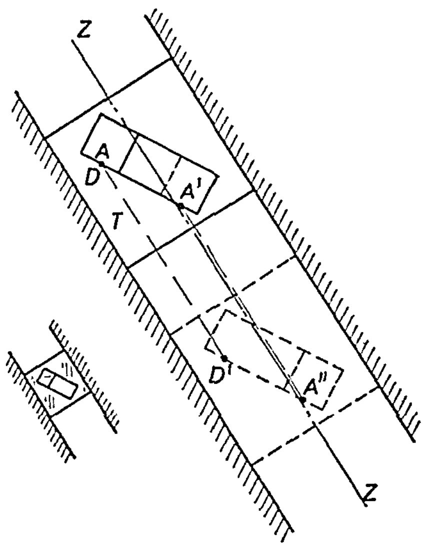


FIG. 95

AA' is still the displacement of A relative to D, but is no longer an absolute displacement, for D is itself now in motion.

The final displacement of A, which is AA'', is the sum of the two changes of position of A, and conforms with Theorem III. Summarizing the motion of A in the terms of the theorem:

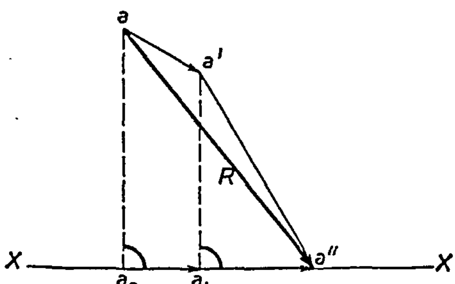


FIG. 96

The resultant absolute displacement of A is the sum of its displacement relative to D plus the absolute displacement of D.

The addition is a vector sum, for displacements are vector quantities. In Fig. 96 the vector solution is shown. While this illustration is at hand, it will be interesting to observe the relationship of the orthogonal components of displacement.

The displacement of A relative to D is aa'. The absolute displacement of D is represented by $a'a''$. Then the resultant displacement of A is aa'',

which by the principle of absolute and relative motion of Theorem III is the absolute displacement of point A.

If we set up any axis, as X–X of the figure, then a_2a'' is the orthogonal component in that inclination of A's absolute displacement.

a_2a'' is the sum of a_2a_1, the orthogonal component of A's displacement relative to D plus a_1a'', which is the orthogonal component in the X-inclination of D's absolute displacement.

The relationship arises directly from the geometry of the vector solution. Applied here, as a corollary to Theorem III, it becomes a basis of first importance. Let us state the conclusion observed from the geometry of Fig. 96, in words.

Theorem III. Corollary 1. The orthogonal component in any given inclination of the absolute displacement of a particle A is the sum of the orthogonal component in that same inclination of the displacement of A relative to another moving particle B, plus the orthogonal component, in that same inclination of the absolute displacement of B.

Expressed as an equation, the corollary appears:

$$s_A = s_{A/B} \nrightarrow s_B$$

in which

s_A is the orthogonal component of A's absolute displacement in a given inclination.

$s_{A/B}$ is the orthogonal component of A's displacement relative to B in the same inclination, and

s_B is the orthogonal component of B's absolute displacement in the same inclination.

VI

VELOCITY

40. Velocity. The preceding discussion of motion has established certain geometrical descriptions associated with change of position, or displacement. In that discussion, we have avoided considering the time which is involved in these changes of position.

The first property of motion which amplifies our knowledge of change of position beyond the geometry of the motion is velocity.

Velocity adds a measure of the time involved in a change of position and is defined as the *rate of change of position with respect to time*, or, more simply, the time rate of displacement.

41. Linear Velocity. When a point or particle changes its position, time is consumed in making that change. During this time, the particle may be moving continuously in one direction as in rectilinear motion, or it may be changing its direction, as in curvilinear motion. In either case, the displacement is a directed or vector quantity, and the velocity, which adds the concept of time, is also a vector quantity, whose direction is identical with the direction of displacement. We must, at the outset, make a clear distinction between the magnitude of velocity, which is called speed, and the fuller description which must include the direction of the velocity.

An automobile's speed may be given as 30 miles per hour. Here we have the description of the amount of changed position with respect to an interval of time, but no information as to the direction. If direction is added, as "30 miles per hour in a northeasterly direction," we have now given the completed description involved in velocity.

When a point travels along a path so that equal amounts of distance are covered in equal intervals of time, the point is said to have uniform motion, or constant speed.

The speed will then be the quotient obtained by dividing any distance, Δs, by the corresponding time interval, Δt.

If the point is travelling along a path which is a straight line, the direction of displacement remains unchanged, and the point has constant velocity.

When the point travels along a curved path, the speed may be constant, but the velocity is changing with the change in direction of displacement and hence is variable. In the case of a point travelling along the circum-

ference of a circle at constant speed, it will traverse equal lengths of arc in equal time intervals, but its velocity will be constantly changing in direction.

When a point travels along a path so that its speed is variable, we must measure its speed instantaneously, since there is no constant or general expression which prevails.

If, in this event, we select a small interval of time Δt during which the point travels a distance Δs, and then set

$$v = \frac{\Delta s}{\Delta t}$$

this value of v will be the average value of speed during the time interval. By decreasing the interval indefinitely, we have

$$v = \underset{\Delta t \doteq 0}{\text{Limit}} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}\cdot$$

Speed is then the first derivative of displacement with respect to time, and we must turn to the calculus for analysis of problems involving variable or non-uniform motion.

As we plan those attacks upon problems in which we must make use of the calculus as a tool of analysis, two avenues of approach confront us.

When displacement-time relationships are available in the form of equations expressing directly the displacement as a function of time, for example, when

$$s = 6\,t^2 + 3\,t + 4$$

the analytical differentiation yields

$$v = \frac{ds}{dt} = 12\,t + 3$$

and speed at any desired time may be computed.

More often, in mechanisms, we encounter displacements which are in such relationships with time that no direct equation, or only an approximate empirical one, may be available.

In these cases, the calculus is again employed, but now, in place of analytical operations, we turn to graphical ones.

PROBLEMS

95. A cutting tool has a length of stroke equal to 4.6 in. If the time per stroke is 0.2 sec., find the average speed of the tool in f.p.m.

96. A mechanism follower travels along a straight path with

$$s = 3\,t^2 + 4\,t$$

where s = displacement in inches
t = time in seconds.

Determine the speed at the end of 4 sec. in f.p.m. *Ans.* 140 f.p.m.

97. A cam follower moves in a straight line with

$$s = t^3 - 8t^2 + 13t$$

with s in feet and t in seconds.

What is the speed of the follower when $t = 0.25$ sec.?

How long does it take the follower to come to rest from $t = 0$?

98. The velocity of a cam follower is

$$v = 1.6t + 0.4$$

with v in inches per second and t in seconds.

How long will it take the follower to travel 5 in. from a start at $t = 0$?

42. Graphical Calculus. Differentiation. The graphical calculus translates, as do many other graphical techniques, a mathematical or analytical procedure into the language of the drawing.

Having noted that speed is the derivative of displacement with respect to time, let us translate the definition of a derivative into graphical terms.

If Δs is any increment (it may be an increase or decrease) given to displacement, and Δt is the corresponding increment in time, then the deriva-

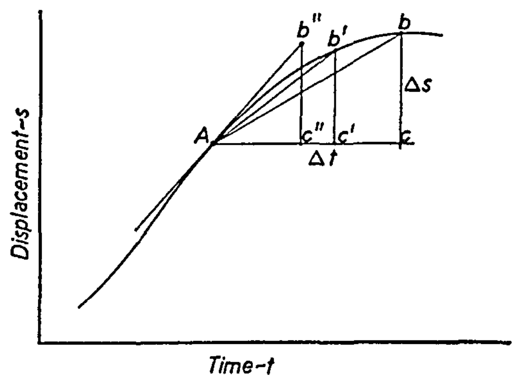


FIG. 97

tive of displacement with respect to time (or speed) is the limit of the ratio of Δs to Δt as Δt approaches zero, or, more tersely,

$$v = \operatorname*{Limit}_{\Delta t \doteq 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

On the curve of Fig. 97 where it is assumed that a curve of the displacement-time relationship is known, it is desired that the speed at any point, as A, of the curve be determined.

$CD = 1.34$ inches, when measured, and $BD = 1.73$ inches.

Then the speed at point A is

$$v_A = \frac{\Delta s}{\Delta t} = \frac{CD}{BD} = \frac{1.34 \text{ actual inches}}{1.73 \text{ actual inches}}$$

$$= \frac{1.34 \times 0.5 \text{ feet (by given scale)}}{1.73 \times 0.01 \text{ seconds (by given scale)}}$$

$$= \frac{0.67 \text{ feet}}{0.0173 \text{ seconds}} = 38.7 \text{ feet per second.}$$

The process of differentiation may be repeated at other points than A and a sufficient number of points obtained to establish a curve showing the speed-time relationship.

A word concerning the number of points required to properly plot a curve is again pertinent here. The infinite variety of shapes of curves makes any fixed rule impracticable. One point orients a straight line whose inclination is also known, two points determine any straight line, and some curves require an extremely large number of points. The rapidity of change of curvature is the only basis upon which intelligent decision may be made, and a sufficient number of points must be taken so that no significant value of the curve is lost. All plotted curves are approximations and the portions of curve lying between plotted points are capable of concealing deviations from the "faired-in" or approximated curves. A safe procedure is to select a large number of points in regions of rapid change, and smaller numbers when the curvature is gradual or slow. Whenever there is suspicion of concealed change, additional points should be taken in the questionable region until possibilities of unusual or unexpected change are exhausted.

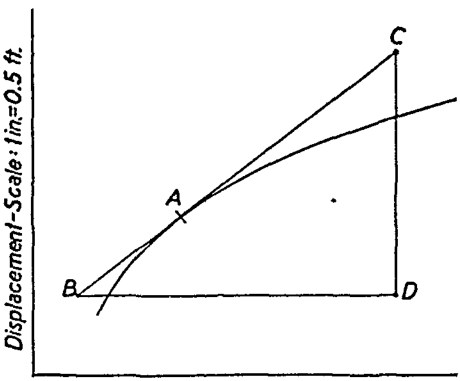


FIG. 99

The process of graphical differentiation is not an exact method, but an approximate one whose potential value is dependent upon the accuracy of setting the tangent. When the method is employed, therefore, it is wise that we have available some means of checking the solution. We shall next turn to another graphical method—graphical integration; which, in addition to serving in its own right, affords a means of checking the graphical differentiation of a given curve.

The area *abcd* represents the integral of the velocity with respect to time for time interval Δt. Since

$$s = \int v dt$$

this area represents the increment of displacement which has been added during this interval. A simple method for determining the amount of area *abcd* is to substitute a rectangle, *mncd*, which is equivalent in area to figure *abcd*. The altitude of the equivalent rectangle is found by moving a straight-edge up or down, always parallel to the axis of abscissae until line *mn* is established at such a level that the small area *bno* appears, by eye, to be equal to area *moa*. In matching these areas, faint trial lines like *mn* are drawn until, by constant trial to reduce error, a final level is established. Here again the trustworthiness of the eye as a judge of matched areas can be relied upon to produce a reasonable accuracy.

With *mn* finally placed, the altitude of rectangle *mncd* multiplied by the base, Δt, gives the increment of displacement. This displacement must be added to the displacement of the body at the beginning of the time interval to obtain the final displacement at the end of time Δt. The initial displacement is thus operating in the same fashion as in analytical integration, where such an initial displacement would appear as the constant of integration; or the graphical integration has given only the change of displacement between the limits of the time interval, Δt.

It will be noted that the accuracy of displacement curve obtained will be increased in proportion to the number of time intervals into which the total motion is divided for study; the approximation of areas by substitution of rectangles is coarse if the time interval selected, Δt, is great, and becomes refined as we operate with lesser time intervals.

Various devices are used to enhance the accuracy of graphical integration—one such is "Simpson's Rule," which may be found in any of the textbooks of the Calculus. In the investigation of mechanisms the drawing method which has been discussed here produces a degree of precision well within the limits of accuracy we demand.

When graphical differentiation is used, the solution rests upon the proper setting of the tangent line. This method is less exact than the addition of areas employed in graphical integration. It is always wise, upon differentiating a kinematic relationship graphically, to apply graphical integration as a check upon the results. When, for example, a displacement-time curve is graphically differentiated, the resulting velocity-time curve should be integrated graphically, and these displacement results checked against the original displacement data.

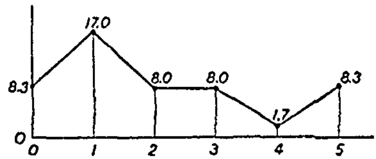
In applying the routine operations of graphical differentiation or inte-

gration it is dangerous to make readings at uniform time intervals unless the curvature of the curve which is being treated is fairly uniform. Instead, in regions where there is rapid change of the character of the curve a greater number of readings should be taken than in regions of more constant nature.

PROBLEMS

105. The curve giving the velocity-time relationship gives values of velocity, v, as ordinates in feet per second, and values of time, t, as abscissae, in seconds. Determine the displacement from O at the end of each second. The portions of the curve are straight lines, and displacements are to be determined, analytically, by calculating the areas.

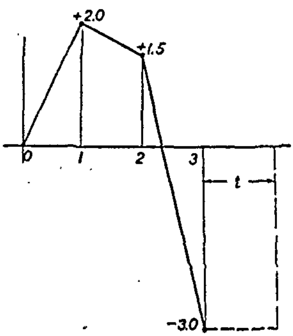
Ans. 12.65 ft. (1 sec.)
25.15 ft. (2 sec.)
33.15 ft. (3 sec.)
38.00 ft. (4 sec.)
43.00 ft. (5 sec.)



PROB. 105

106. A particle travels in a straight line, starting at an origin O, with the velocity-time curve shown. Determine, analytically, the displacement at the end of each second. Velocities (ordinates) are given in feet per second, and time intervals (abscissae) in seconds. Velocities plotted as positive values are of one sense, those plotted as negative values are of opposite sense.

How long (time t) will it take the particle to return to the origin if it continues to travel at constant velocity of 3.0 f.p.s. after the 3-sec. mark?



PROB. 106

Problems 107–109. Given a table showing the velocity-time relationship for a moving particle whose path is a straight line. Plot the curve, and integrate graphically. Plot the displacement-time curve.

107.

Velocity	*Time*
0 in. per sec.	0 sec.
1.91	1
3.58	2
4.92	3
5.76	4
6.07 (max.)	5
5.89	6
5.30	7
4.47	8
3.44	9
2.31	10
1.16	11
0	12

108. *Note: Velocities of one sense are labelled positive; those of opposite sense negative:*

Velocity	*Time*
0 in. per sec.	0 sec.
+0.30	0.25
+0.54	0.50
+0.80	0.75
+1.01	1.00
+1.17	1.25
+1.29	1.50
+1.36	1.75
+1.40	2.00
+1.40	2.25
+1.39	2.50
+1.34	2.75
+1.27	3.00
+1.15	3.25
+0.94	3.50
+0.59	3.75
0	4.00
−0.92	4.25
−2.16	4.50
−3.55	4.75
−4.19	5.00
−3.12	5.25
−1.48	5.50
−0.49	5.75
0	6.00

109. *See Note of Problem 108.*

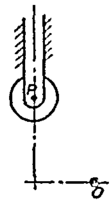
Velocity	*Time*
0 in. per sec.	0 sec.
+0.45	0.3
+0.79	0.6
+1.09	0.9
+1.37	1.2
+1.64	1.5
+1.89	1.8
+2.13	2.1
+2.33	2.4
+2.48	2.7
+2.55	3.0
+2.54	3.3
+2.40	3.6
+2.12	3.9
+1.69	4.2
+1.04	4.5
0	4.8
−0.45	4.9
−0.97	5.0
−1.63	5.1
−4.04	5.4
−5.00	5.5
−5.94	5.6
−6.70	5.7

Velocity	*Time*
−7.22 in. per sec.	5.8 sec.
−7.32	5.9
−6.96	6.0
−6.28	6.1
−5.32	6.2
−4.39	6.3
−2.03	6.6
−0.70	6.9
0	7.2

110. A cam follower is to be of the type shown. The cam makes one revolution in 8 sec., clockwise, about fixed axis O. The values of the displacement schedule for the first 4 sec. are those values obtained in the first 4 sec. portion of Problem 108. During the remaining 4 sec. the follower is to remain at rest, and then return at once to its starting position, which is point P. $OP = 2.0$ in. Design the plate cam.

PROB. 110

111. A cam follower is of the type shown. The cam makes one revolution in 7.2 sec., clockwise, about fixed axis O. The starting position of the center of the roller follower is at point P. Point O is 1.4 in. below, and 1.0 in. to the right of P. The values of the follower velocity for the first 6 sec. are those given in Problem 109. The follower then dwells for 1.2 sec., and next returns immediately to its starting position. Design the plate cam.



PROB. 111

44. Angular Velocity is the *time rate of angular displacement.* Here again we may have uniform motion, and the angular velocity ω is the quotient obtained by dividing the angular displacement $\Delta\theta$ during a time interval, by that time interval

$$\omega = \frac{\Delta\theta}{\Delta t}.$$

If the angular velocity is variable, we must, as in the case of the linear counterpart, deal with instantaneous values by averaging the velocity over an increasingly smaller interval of time, until in the limit we establish the instantaneous value

$$\omega = \underset{\Delta t \doteq 0}{\text{Limit}} \frac{\theta}{\Delta t} = \frac{d\theta}{dt}.$$

All velocities are vector quantities—they possess magnitude, inclination, and sense.

In representing angular velocity, a vector quantity, by its graphical translation into a vector, we adopt conventions to indicate magnitude, inclination, and sense.

The *magnitude* of the vector quantity becomes, as in our other cases, the length, by assigned scale, of the drawn vector.

The *inclination* means, in this case, a description of the plane of the rotation. We must announce with the drawing of the vector the plane in which the body has angular velocity. This is indicated by erecting the vector perpendicular to the plane of rotation.

Sense of the vector quantity announces whether the angular velocity is clockwise or counter-clockwise in the plane of rotation. This is indicated by

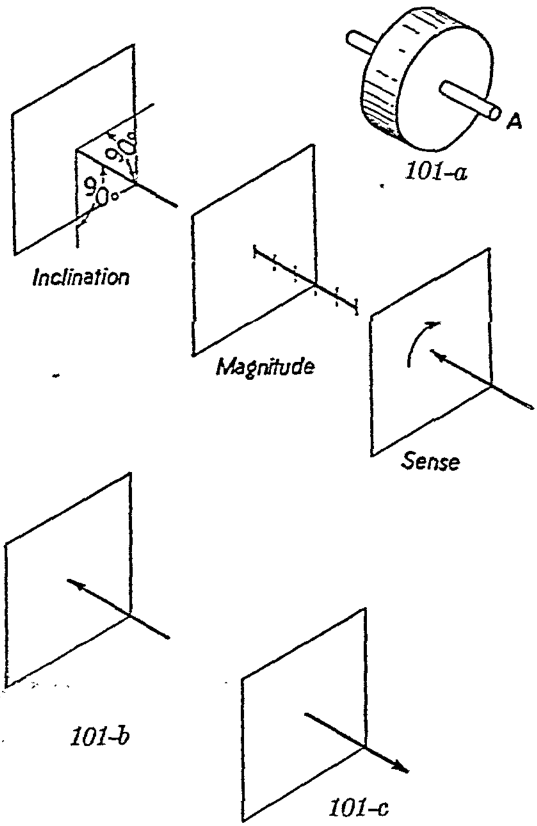


Fig. 101

placing an arrow-head at the end of the vector, so that as we look from origin to terminus we will observe clockwise rotation.

A definite application will make these conventional steps of representation clearer.

A cylinder, shown in Fig. 101-a, is mounted upon a horizontal shaft. It has a speed of 5 radians per second, clockwise as we look from end A of the shaft toward the cylinder. The vector which will represent the angular velocity of the cylinder is to be drawn.

We note that the plane of rotation is vertical, and erect a line, normal to the plane of rotation, or horizontal. This is customarily placed along the axis of rotation of the rotating body. We have now announced the inclination of the vectorial angular velocity, or the plane in which rotation is taking place.

We next lay off, to any convenient scale, a distance along the vector to represent the magnitude of 5 radians per second.

Finally, we announce that the sense of the velocity is clockwise by so placing an arrow-head that as we look along the vector from origin to terminus we are observing clockwise rotation. The completed vector is shown in Fig. 101-b.

If the description had been as follows: The body has angular velocity equal to the previous one in every respect except sense, which is counterclockwise when we look at the cylinder from end A of the shaft, we should find, as in Fig. 101-c, a vector, identical with the previous one in every respect except that the arrow-head is placed as shown.

The graphical representation of angular velocity as a vector is much less frequently encountered than that of linear velocities, but is an equally efficient tool. When the representation has been accomplished, these vectors are analyzed as are all vectors—the same rules of procedure apply in resolution or composition. A body having an angular velocity in a given plane has component angular velocities in planes oblique to the plane of rotation. Then the vector representing its resultant angular velocity may be resolved into components representing the component angular velocities in the oblique planes. Or, again, if a body is found to be rotating relative to one axis while that axis is itself rotating relative to a second axis, the component angular velocities may be represented by vectors and these vectors added to obtain the resultant angular velocity of the body.

45. Relation between Linear and Angular Velocity. Point B in Fig. 102 is moving as line AB rotates about axis A.

The linear speed of point B at any instant is $v = \dfrac{ds}{dt}$, where ds is the linear displacement in time dt.

During this same interval of time the line AB undergoes angular displacement $d\theta$.

The linear displacement ds is equal to $r d\theta$ (Art. 33).

Then

$$v = \frac{r d\theta}{dt}.$$

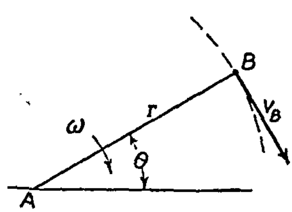


FIG. 102

113. A twist drill, $\frac{3}{8}$ in. in diameter, has an angular velocity of 300 r.p.m. Calculate the speed of a point on the surface of the drill in f.p.s.

114. A point on a wheel rotating about its own axis has a speed of 1500 f.p.m. If the point is 6 in. from the axis, calculate the angular velocity of the wheel and the speed of a point 8.6 in. from the axis.

115. Two cylinders are keyed to the same shaft. If the surface speed of cylinder A is 2400 f.p.m., calculate the surface speed of cylinder B. The diameter of cylinder A = 3.40 ft., and the diameter of cylinder B = 1.92 ft. *Ans.* 1355 f.p.m.

116. A gear has an angular velocity of 500 r.p.m. If a point A on the gear has a speed of 4000 in. per min., how far is point A from the axis of the gear?

117. Two points, A and B, on a radius of a pulley have speeds of 1500 and 2000 f.p.s., respectively. If the angular velocity of the pulley is 1850 r.p.m., how far apart are the points?

46. Graphical Analysis of Velocity Vectors. In graphical solutions, ratios and proportions are established through the medium of similar triangles.

For example, if the linear velocity of a point A, Fig. 104, is known, the

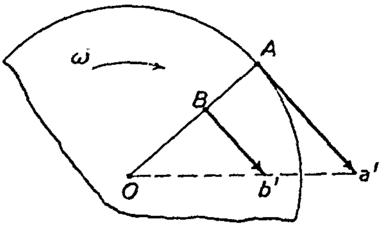


FIG. 104

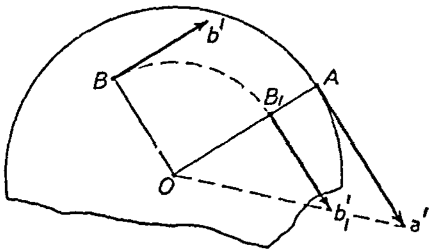


FIG. 105

velocity of any other point, like B, may be found by constructing similar triangles to yield the proper proportion.

Let the known velocity of A be Aa' (equal to angular velocity of the rotating body, ω, times the radius OA).

Draw line $a'O$ from the terminus of A's velocity to the axis of rotation, O.

Erect, at origin B, a vector whose direction is perpendicular to radius OB. The intersection of this vector with $a'O$ will be its terminus, and Bb' is the linear velocity of point B.

We note that Bb' is in the correct ratio with Aa'—the two velocities are to each other as the distances from center O, since triangles OBb' and OAa' are similar. Further, the directions of both are perpendicular to the moving radial line, which conforms with the definition of direction of linear velocity.

When the points A and B do not lie on the same radial line, we have the condition illustrated in Fig. 105.

Given the linear velocity of point A as Aa', it is desired that the linear velocity of point B be found.

Velocity has two properties—magnitude or speed, and direction.

The magnitude of B's velocity may be obtained by swinging an arc of radius OB and center O until it meets radius OA at B_1.

The speed of point B_1 is obtained as in the previous case, and is B_1b_1'. This quantity is also the speed of point B, since points which are equally distant from center O will have the same speed. Returning now to point B we erect a vector perpendicular to OB, which establishes the direction of

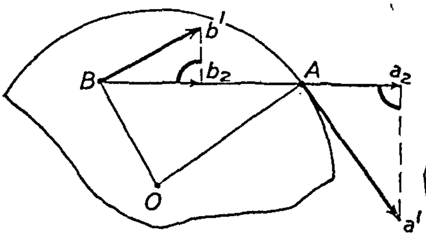


Fig. 106

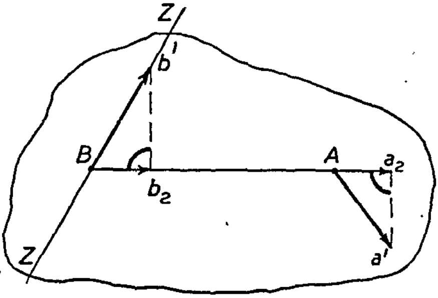


Fig. 107

B's velocity, and of length equal to B_1b_1' which establishes its magnitude. *The completed vector Bb' is the linear velocity of point B.*

Additional information on the relationship of the linear velocities of points on the same body may be obtained by recalling the concept of the rigid bodies of mechanics. These have already been described as non-deformable. If, for example, the wheel of Fig. 106 is not deformed during its motion, points O, A, and B will remain constantly at the same distance from each other.

Since the distance AB remains unchanged during any motion, it follows that points A and B must have the same orthogonal component of velocity in the direction $AB(Bb_2 = Aa_2)$. We can readily prove this to be true by noting that if B has a greater or lesser orthogonal component of velocity in the direction AB than point A, it would be drawing nearer to or receding from point A, and the distance AB would be a changing one.

We conclude that any two points on a rigid body have the same orthogonal component of linear velocity in the direction connecting the two points.

This relationship is of value. Together with the theorems of orthogonal components (Art. 12) we have an additional weapon of analysis for attacking problems of linear velocity.

Application of Theorem I. Given, as in Fig. 107, two points A and B which lie on the same rigid body. Given also the linear velocity of point A, as Aa', *and a known inclination of the velocity of point B*, as the direction ZZ. It is desired to find the linear velocity of point B.

We first find the orthogonal component of A's linear velocity in the direction AB. This is Aa_2.

Next the orthogonal component of B's velocity in direction AB, Bb_2, is made equal to Aa_2.

We now have, for point B, one orthogonal component and a known inclination of resultant. By erecting b_2b' perpendicular to AB to intersect known direction ZZ we obtain Bb', the linear velocity of point B.

Application of Theorem II. We are given, as in Fig. 108, three points A, B, and C which lie on the same rigid body, and hence are at fixed distances, AB, BC, and CA from each other. The velocity of point A is Aa', which is known. The velocity of point C is Cc', which is also known. We are to find the velocity of point B.

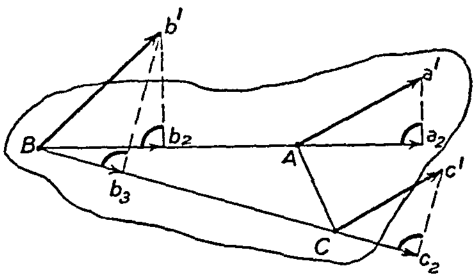


FIG. 108

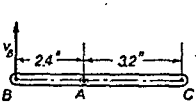
Points A and B must have the same orthogonal component in direction $AB(Aa_2 = Bb_2)$.

Similarly, points C and B must have the same orthogonal component in direction $CB(Cc_3 = Bb_3)$.

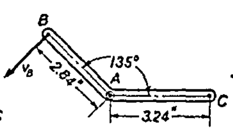
Then two orthogonal components of B's velocity are known, and the resultant linear velocity Bb' is obtained by erecting perpendiculars to Bb_2 and Bb_3 at their termini, which will intersect at b', which in turn becomes the terminus of the resultant velocity of point $B(Bb')$ by the principle summarized in Theorem II.

PROBLEMS

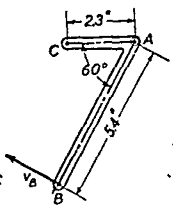
Problems 118–121. The body shown in the figure rotates about a fixed axis at A. If the velocity of point B is 2 in. per sec., find the velocity of point C, graphically. Calculate the angular velocity of the body in r.p.m.



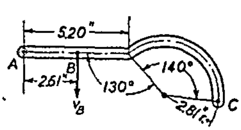
PROB. 118



PROB. 119



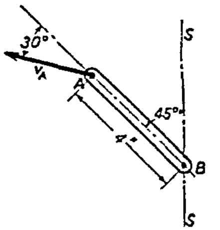
PROB. 120



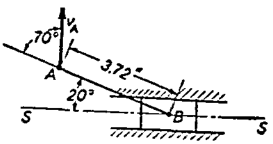
PROB. 121

Problems 122–125. The velocity of point A is 25 f.p.m. The inclination of B's velocity is known and is given as the axis S–S. Find, graphically, the magnitude and sense of B's velocity by applying Theorem I.

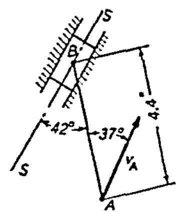
126. The velocity of point A is $v_A = 50$ f.p.m. Find the surface speed of the wheel, W, graphically, and calculate its angular velocity in r.p.m. The diameter of the wheel is 5.0 in. *Ans.* 51 f.p.m.; 39 r.p.m



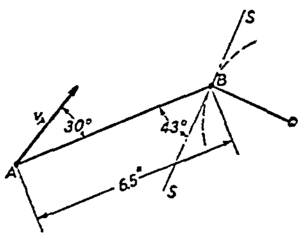
Prob. 122



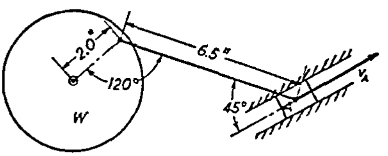
Prob. 123



Prob. 124



Prob. 125

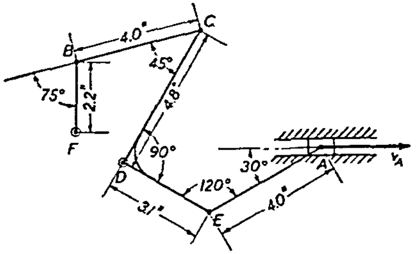


Prob. 126

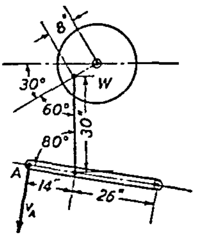
127. In the series of links shown, point A has a velocity of 200 f.p.m. Find the velocity of crank BF. Links CD and DE are fastened to each other and form one rigid body.

128. Point A of the treadle has a velocity of 65 f.p.m., in the position shown. Find the angular velocity of the grindstone, W, and its surface speed. The diameter of the grindstone is 2 ft. *Ans.* 11.5 r.p.m.; 72 f.p.m.

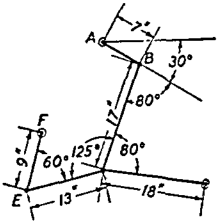
129. The toggle mechanism shown consists of a series of links fastened to each other by pins. If the angular velocity of crank AB is 2 radians per sec., counter-clockwise, determine the angular velocity of crank EF, at the position shown.



Prob. 127



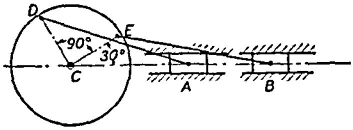
Prob. 128



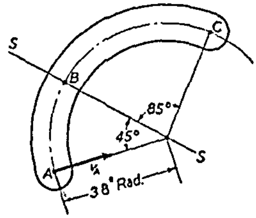
Prob. 129

130. In the engine shown, piston A has a velocity $v_A = 300$ f.p.m. to the right. Find, graphically, the velocity of piston B. Cranks CD and CE are 4.5 in. long, and connecting rods DA and EB are 12.5 in. long.

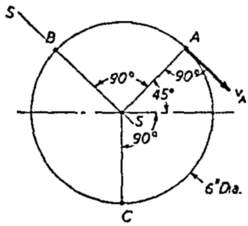
Problems 131–134. The velocity of point A and the inclination of the velocity of point B are known. $v_A = 2$ in. per sec. The inclination of B's velocity is the axis S–S. Find the resultant velocity of point B by applying Theorem I, and the velocity of point C by applying Theorem II. The outline of the machine part containing points A, B, and C is shown in the figure but need not be drawn in the graphical solution.



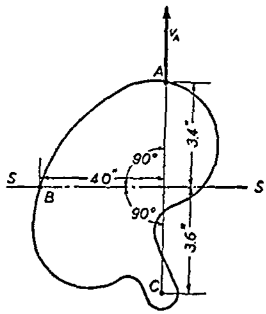
Prob. 130



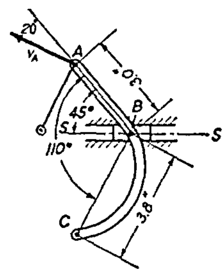
Prob. 131



Prob. 132



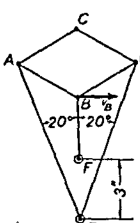
Prob. 133



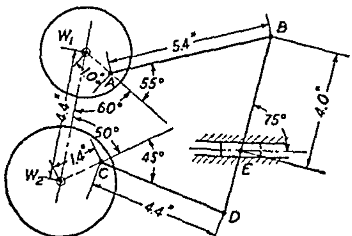
Prob. 134

135. Point B of a Peaucellier's Cell Linkage has a velocity of 3 in. per sec. Find the velocity of point C. The cell is composed of four equal links AB, BD, DC, and CA. Cranks AE and DE are 8.1 in. long. Crank BF is 3 in. long. *Ans.* 4.61 in. per sec.

136. Wheels W_1 and W_2 have an angular velocity of 2 radians per sec. counter-clockwise. The arm BD is attached to W_1 by connecting rod BA, and to W_2 by connecting rod DC. BED is one continuous member. Point E of arm BD is pinned to a block which slides in fixed guides. Find the velocity of point E.



Prob. 135



Prob. 136

47. The Instantaneous Axis of Velocities. When a body like BC of Fig. 109 has a determinate motion, the various particles of the body must have motion which is in some definite relationship.

The body BC is, from the definition of the rigid body of mechanics, unlimited in extent. Points which lie directly along the line BC all have resultant velocity of some finite magnitude. If we leave the line BC, but are careful to deal always with points of the body BC, we can find a point which has, at this instant, zero resultant velocity. Since B and C rotate in circular paths about A and D, respectively, the inclinations of their resultant velocities are known.

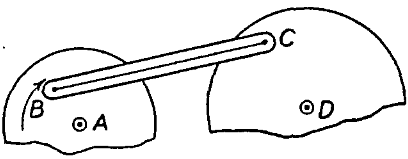


Fig. 109

In Fig. 110 the location of the point on body BC which has zero velocity has been determined. A line BI perpendicular to Bb', the resultant velocity of point B, and another line CI perpendicular to Cc', the resultant velocity of point C, are drawn. These lines intersect at point I which will be called the *instantaneous center* of body BC.

An axis through this instantaneous center, perpendicular to the plane of motion, will be called the *instantaneous axis of velocities*, or, more briefly, the instantaneous axis.

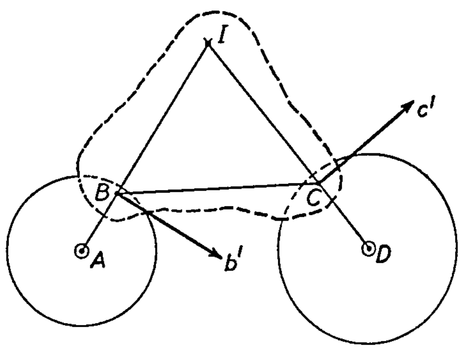


Fig. 110

This point I has zero velocity.

Since points B and I lie on the same rigid body BC, the orthogonal components of velocity in the direction IB connecting them must be the same. But the resultant velocity of point B is perpendicular to IB, and has no orthogonal component in this direction. If I has any resultant velocity, that velocity must then be perpendicular to IB. Then if I has any velocity at all, that velocity must have an orthogonal component in the direction IC; and point C, on the same rigid body as I, will have the same orthogonal component in the direction IC.

But we now note that IC is perpendicular to Cc', the resultant velocity of point C. C cannot have an orthogonal component of velocity in this direction. It follows that I cannot have such an orthogonal component either. Then I must have zero velocity.

A wheel rotating about its own center as a fixed axis is an example of pure rotation. This axis is also a point of zero velocity. In the present case,

we have reproduced the motion of pure rotation by finding a point on the body BC which *at the instant* has zero velocity, and hence is acting, just as in the case of the wheel, as an axis of rotation. There is a vital distinction between the two cases which must be observed. In the case of the wheel in pure rotation, the zero-velocity axis is a *fixed* or permanent axis. In the case of body BC, the zero-velocity axis is not fixed but is operative as an axis of rotation *at the given instant only.*

We can reinforce this thought by noting that when a body BC is displaced from the position shown in Fig. 111 to a new position as in Fig. 112 particle I

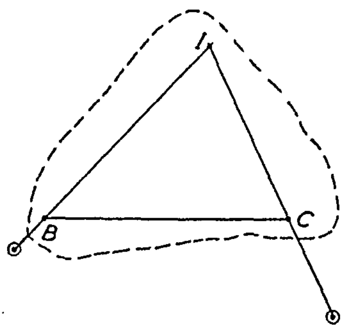


FIG. 111

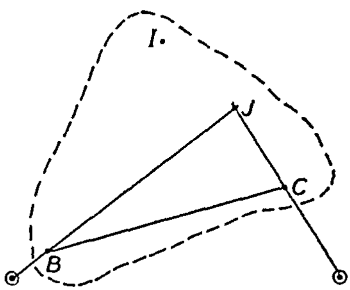


FIG. 112

of body BC is no longer at the intersection of lines perpendicular to the resultant velocities Bb' and Cc', but is elsewhere on the body, and a different particle of the rigid body, namely J, is now at the intersection, and may be shown as before to be the point of zero velocity, or the instantaneous center for the new position of BC.

When a body is moving in plane motion, then, we may consider it, *at any given instant and at that instant only,* to be moving in pure rotation about an instantaneous axis of velocities. We may here as in any case of pure rotation apply the principles of velocity relationship between points on the same rotating body.

Illustrative Example. Given the mechanism shown in Fig. 113, with the angular velocity ω_1 of wheel W_1 known. We are to locate the instantaneous axis of body BC, and then to find the velocity of any point which, like point S, lies on body BC. Wheel W_1 is rotating about a fixed axis A, and wheel W_2 is rotating about a fixed axis D.

The direction of B's resultant velocity is fixed, since B must travel in a circular path about point A and its resultant velocity is therefore perpendicular to AB. The direction of C's resultant velocity is also known, and is perpendicular to CD. If we erect perpendiculars to these resultant velocities (these perpendiculars are, of course, extensions of AB and CD)

their intersection I_{BC} is the instantaneous center of BC. At this instant the entire body BC has a motion of pure rotation about I_{BC}.

Point S is moving, at this instant, in a circular path about I_{BC}, as is every other point on body BC. Then the direction of the resultant velocity

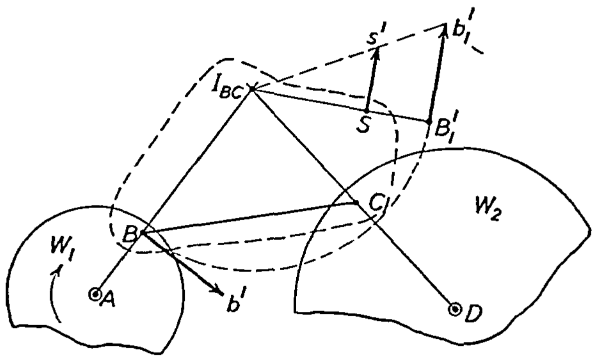


Fig. 113

of S is perpendicular to $I_{BC}S$. We can establish the magnitude of S's resultant velocity by proportion. The point B has resultant velocity $Bb' = \omega_1 \cdot AB$. The magnitude of Bb' is used to establish the magnitude of S's velocity by similar triangles as in the previous analyses of rotation, and Ss' is obtained as the resultant velocity of point S.

We have available other methods for finding such velocities as that of point S. For example, we may turn to Theorem I and establish, as in Fig. 114, the orthogonal component Ss_2 of S's velocity in the direction of BS.

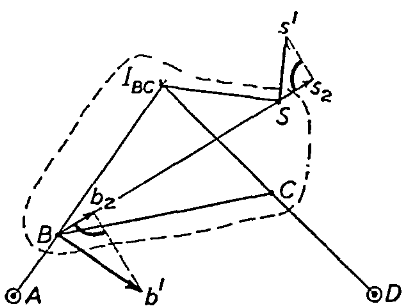


Fig. 114

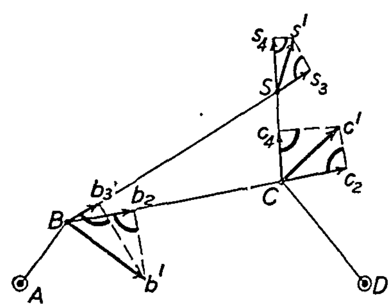


Fig. 115

The instantaneous axis I_{BC} of body BC may then be used to fix the inclination of S's resultant velocity. Having one orthogonal component, and the inclination of the resultant velocity, the resultant velocity is determined.

Or, again, we may resort to Theorem II, as in Fig. 115.

Bb' is known and Cc', the resultant velocity of C, may first be determined

by setting $Cc_2 = Bb_2$, the orthogonal component in direction BC, and establishing C's direction of velocity as perpendicular to CD, since C travels in a circular path about fixed axis D. This is another application of Theorem I.

With two resultant velocities Bb' and Cc' now known, we observe that S has orthogonal component $Ss_3 = Bb_3$ in the direction BS, and $Ss_4 = Cc_4$ in direction CS. This procedure yields two known orthogonal components of the resultant velocity of point S, and, by Theorem II, the resultant velocity of $S = Ss'$ is now determined.

The three methods of attack outlined above yield equivalent results. It is desirable to take advantage of more than one method since opportunity is afforded to check the results. Apart from the mental stimulus we feel in accomplishing a valid check of our work, we go forward into use of these procedures in further analysis with the confidence that they have been proven trustworthy. In addition, a multiplicity of instruments of analysis enables us to face a broader range of possible situations than would be possible with more limited equipment.

Two more examples of plane motion are frequently encountered. These follow, and will be pursued only to the point of locating the instantaneous axis, at which time the method of solution merges with the method outlined above.

Let us first summarize the equipment needed thus far for the location of an instantaneous center. The instantaneous center of a body may be determined when the inclinations of the resultant velocities of two points are known; provided that these resultant velocities are not parallel. Let us now see what material is necessary when two points of the body do have parallel velocities.

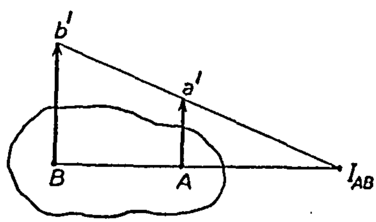


Fig. 116

In Fig. 116, body AB is moving so that velocities Bb' and Aa' are parallel. Here we find that we must know the magnitudes of the velocities in addition to their directions, in order to locate the instantaneous center. With magnitudes known, we may draw a straight line through b' and a', meeting AB produced at I_{AB} which is then the instantaneous center of the body AB. The resultant velocities of A and B are both perpendicular to the radius from the instantaneous center, and the magnitudes of these velocities are in direct proportion to the distances $I_{AB}A$ and $I_{AB}B$ from the axis to the respective points.

When a body is constrained to move with translation, all points have equal and parallel velocity (Fig. 117).

Then perpendiculars to the individual velocities are parallel, and will not converge.

The instantaneous axis must in that event be located at an infinite distance from the body.

A word of caution may be quite proper here, even though it introduces no new concept but insists only upon careful attention to those already developed.

The instantaneous center of velocity of a body is a point which definitely belongs to that rigid body. When velocity analyses of connected bodies

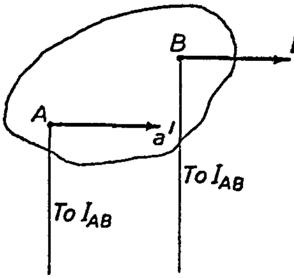


Fig. 117

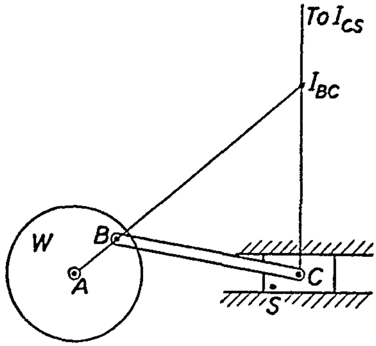


Fig. 118

are made, we must assign each instantaneous axis to the particular body to which it belongs and to no other. In this text each instantaneous axis is identified by denoting the body to which it belongs as a subscript; e.g., I_{BC} signifies the instantaneous axis of body BC.

In Fig. 118, a series of connected bodies is shown.

The wheel W is moving with pure rotation about an axis which is permanently fixed at point A. This is an example of an axis of rotation which remains the same at all instants. It is a point of body W, and serves as the rotational center for all resultant velocities of points on W.

Connecting rod BC has plane motion, with its instantaneous axis, for the instant shown, at I_{BC}, which is a point of body BC and the resultant velocities of points on BC have their axis here.

Finally, sliding block CS has a motion of pure translation, with its instantaneous axis, I_{CS}, at infinity.

PROBLEMS

Problems 137–139. Locate the instantaneous axis of velocities of body AB, graphically.

140. Locate the instantaneous axis of body AB, Problem 122.

141. Locate the instantaneous axis of body AB, Problem 123.

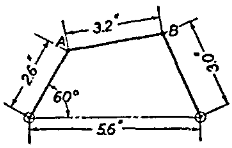
142. Locate the instantaneous axes of bodies BC and EA, respectively, Problem 127.

143. Locate the instantaneous axes of bodies EB and DA, respectively, Problem 130.

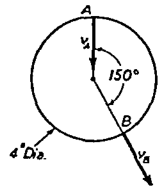
144. The velocity of point A is 2 f.p.m., and the velocity of point B is 3.2 f.p.m. Locate the instantaneous axis of body AB.

145. The velocity of point A is 1.54 in. per sec. Point B has a velocity of 2.32 in. per sec. Locate the instantaneous axis of body AB.

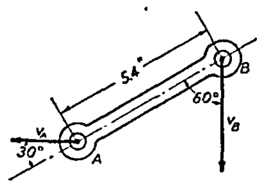
146. The angular velocity of the body BC is 4 radians per sec. Find the angular velocities of cranks AB and CD.



PROB. 137



PROB. 138



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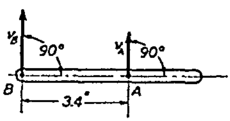
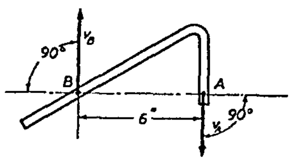
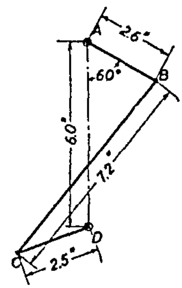


PROB. 144



PROB. 145



PROB. 146

48. Velocity Analysis of Plane Motion. Combined Translation and Rotation.

The instantaneous axis has furnished one road of attack on the anal-

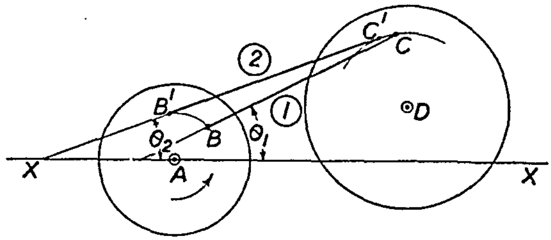


FIG. 119

ysis of plane motion. Of equally important fundamental value is the division of such motion into its two component bases—translation and rotation.

Let us therefore explore the mechanism shown in Fig. 119 where body BC illustrates a typical case of plane motion. We note that if we rotate the wheels from position (1) to position (2) the inclination of the line BC (note the angle which it makes with the X-axis) has changed. Then this motion cannot be one of pure translation. Next, we note that while point B is

turning about a fixed axis A, point C is turning about fixed axis D. Then the motion cannot be pure rotation, which demands that all points in BC be turning about the same fixed axis.

We may, however, analyze the motion by "breaking it down" into two parts.

We can reproduce the resultant displacement of link BC from position (1) to position (2) by travelling in two successive stages, as in Fig. 120.

First Stage: Rotate line BC about any point, like R, in BC (or BC produced) until it lies parallel to the final position $B'C'$. This intermediate position is $B''C''$. During the time interval of this first stage, the motion has been one of pure rotation since the line has rotated about a fixed axis.

Second Stage: Now move the line so that B'' goes to B' and C'' goes to C'. This motion is one of pure translation, since the line is remaining con-

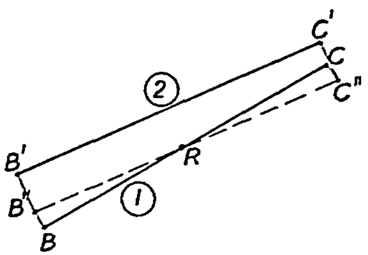


FIG. 120

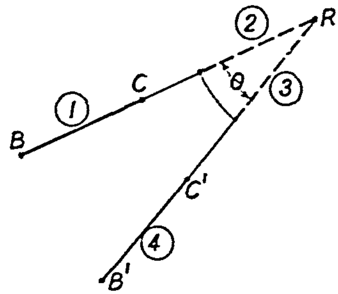


FIG. 121

stant in its inclination while moving to the final position, and all points are travelling in parallel paths.

We have now established a most useful division of the two elements in a plane motion. We have found that any plane motion of a body may be divided into two stages—a rotation of the body about any point plus a translation of that point (and hence of the entire body). The sum of these two stages is equivalent to the resultant plane motion, and the division frequently forms a more convenient method of attack than dealing with the resultant motion as a single stage. This type of analysis is usually described as "Combined translation and rotation."

In the previous examination of the nature of plane motion, we have concerned ourselves with viewing only the two stages of displacement which are involved.

Let us now analyze the factors of velocity which make up the plane motion.

A slightly more elaborate path of travel will be useful here. Let the line BC (Fig. 121) be moved from starting position (1) to final position (4) or $B'C'$ as follows:

(1) Move line BC along its own path produced until point C is at R, the intersection of BC and $B'C'$. The line is now in position (2). This element of motion has been pure translation.

(2) Now rotate the line from position (2) about R as an axis through angle θ, until it coincides in direction with $B'C'$. The line is now in position (3). This motion has, of course, been pure rotation.

(3) Finally, move the line along $B'C'$ produced until it occupies the desired final position (4). This has again been a motion of pure translation.

Let us analyze the velocities involved in the complete motion.

During (1) all points of BC have a velocity whose direction is along BC, and these velocities are all equal.

During (2) all points of BC have velocities which are constantly perpendicular to BC and in the ratio of their respective distances from R.

During (3) all points of the line will again have velocities in the direction of the line and these velocities will again be equal to each other.

Now we shall consider increasingly smaller amounts of angular displacement θ. As the angle becomes smaller, position (3) approaches position (2) as a limit. The linear velocities of the line approach perpendiculars to position (1) of BC as their limiting direction. In the limit the linear velocities of the rotation are perpendicular to BC, and their magnitudes are proportional to their respective distances from R.

At the same time the equal velocities involved in the translation from position (3) to position (4) are approaching in direction the line BC itself. In the limit, all of the translation velocities will be in direction BC.

Now, the approach which we have made to a limit has really been an approach which has established the nature of the velocity of the points in line BC at any instant in a plane motion. It reveals the fact that the instantaneous velocity of such points has two components; a *component of translation*, which lies along the line itself, and a *component of rotation* which is perpendicular to the line. The components of translation of all points have equal magnitudes, the components of rotation are proportional in magnitude to their distance from some point which, like R of Fig. 121, is serving as an axis of rotation. The components of translation and rotation are mated rectangular components.

We have added another means of analysis to our growing store. Let us apply this principle to the example which is shown in Fig. 122.

At the instant, point B has known velocity Bb'. Then C's velocity Cc' may be found, as previously, through combining one known orthogonal component Cc_2 $(=Bb_2)$ and known inclination—perpendicular to CD. It is desired that we find the velocity of point E, a point on body BC.

Body BC has plane motion. At the instantaneous position shown, all

points in BC have components of translation along BC. Bb_2, the orthogonal component of Bb' in direction BC, is one such component of translation and E must have the same component of translation. This is established as $Ee_2 = Bb_2$.

B's component of rotation must be rectangular component Bb_3 since that is the mated rectangular component perpendicular to BC. C's component of rotation is Cc_3. We know that these components of rotation are in direct proportion to their distances from an axis of rotation, and can again resort to similar triangles to set the proportion graphically. We

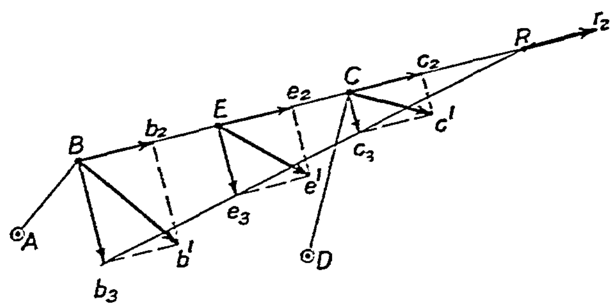


Fig. 122

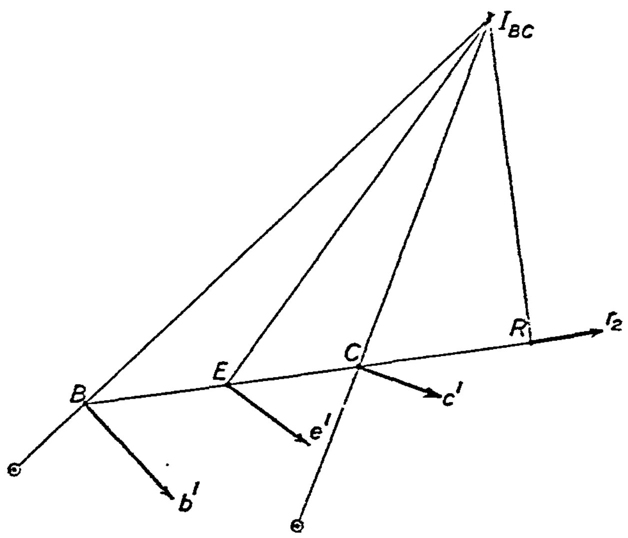


Fig. 123

therefore draw a line from terminus b_3, through c_3 to intersect BC produced at point R. Then we erect a vector perpendicular to line BC at E, having its terminus e_3 in line b_3c_3R. Then Ee_3 is E's component of rotation, for

$$\frac{Bb_3}{Ee_3} = \frac{BR}{ER}.$$

We now have Ee_2 as component of translation, Ee_3 as component of rotation, and may add them to obtain vector Ee', which is the desired resultant velocity of point E.

We should note, before leaving this example, that point R itself has velocity. Since it lies along the line BC it must have the same component of translation which points like B, C, and E have. Then R's component of translation is $Rr_2 = Bb_2$ or Cc_2 or Ee_2. But R is also the center or *axis of the components of rotation*, and has therefore no component of rotation. Rr_2 is, then, the resultant velocity of point R.

The resultant velocities of points B, E, C, and R are all resultant velocities of points of one body which is in plane motion. They may be checked therefore, as in Fig. 123, by finding the instantaneous axis of the body BC. I_{BC} will lie at the intersection of line $I_{BC}B$ and $I_{BC}C$ drawn perpendicular to Bb' and Cc' respectively. If now lines $I_{BC}E$ and $I_{BC}R$ be drawn, they must be perpendicular to Ee' and Rr_2 respectively.

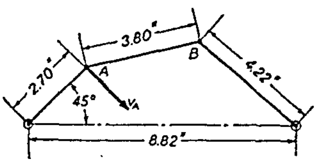
PROBLEMS

Problems 147–149. Find the components of translation and rotation of points A and B along and perpendicular to line AB. Locate R, the axis of the components of rotation, and determine its velocity. Resultant velocity of A = 3 in. per sec.

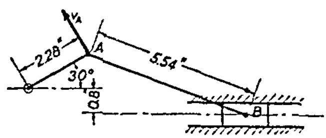
Problems 150–151. Determine the velocity of point E, given the velocity of point A = 2 f.p.m.

(1) Divide the plane motion of the body containing E into components of translation and rotation.

(2) Check results by using the Instantaneous Axis method.



PROB. 147



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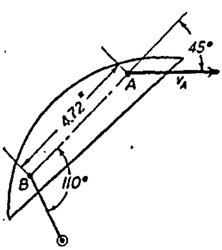

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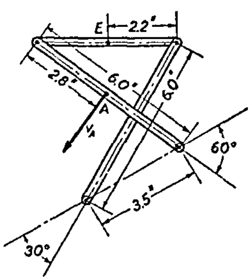
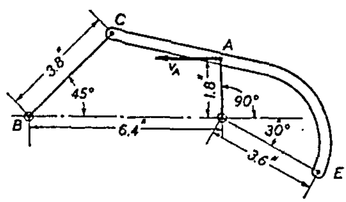


PROB. 150



PROB. 151

152. Using the data of Problem 139, determine the velocity of a point E, the mid-point of line AB, by the components of translation and rotation.

49. Absolute and Relative Velocity. In Chapter IV general concepts of absolute and relative motion were developed. Velocity, a property of motion, may be classified as absolute or relative, depending, as in the general motion definitions, upon the choice of reference point or axis.

When a point or axis fixed to the earth is used as reference, velocity becomes *absolute velocity.*

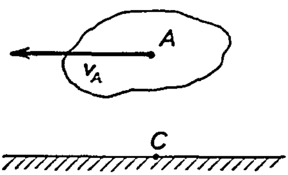


FIG. 124

When the velocity of one point is referred or related to any point other than one fixed upon the earth's surface, that velocity is a *relative velocity.*

If, as in Fig. 124, a body is moving so that the velocity of point A, relative to point C, which is a fixed point on the earth, is v_A, then v_A is an absolute velocity.

We have already noted (Theorem III) that the absolute motion of a point may be analyzed by the less direct method of relating it first to a second point, serving as a reference, and then considering the motion of the second point relative to the earth.

An illustration in the form of rectilinear motion will furnish a simple exposition of absolute and relative velocity and help in crystallizing our grasp of their relationship.

Car A in Fig. 125 has a velocity at the instant of observation of 20 miles

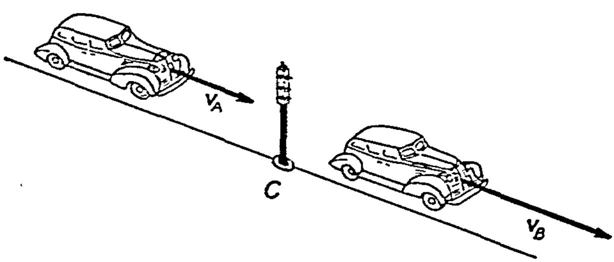


FIG. 125

per hour to the right. This is its absolute velocity and car A will approach point C, a fixed point on the road, at the rate of 20 miles per hour.

Car B has at the same instant an absolute velocity in the same direction of 45 miles per hour and is therefore travelling away from point C at that rate.

The difference between B's velocity and A's velocity is the velocity of B relative to A or 25 miles per hour, to the right.

This difference is a difference between two vector quantities and may

be obtained by the method of subtracting vectors outlined in Art. 7. $v_{B/A} = v_B \rightarrow v_A$. The vector solution is shown in Fig. 126.

It will be noted that in obtaining the velocity of B relative to A we subtract the velocity of the point to which we are referring as an axis from the velocity of the point we are relating to it. This order is important, since the vector solution must be relied upon to fix the sense of the relative velocity as well as its magnitude and inclination.

A further illustration will reveal the necessity of using care in the order of setting the terms in the subtraction operation. The velocity of "A relative to B" indicates that we are interested in subtracting B's absolute ve-

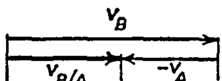


FIG. 126

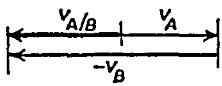


FIG. 127

locity from A's absolute velocity. $v_{A/B} = v_A \rightarrow v_B$. Figure 127 is the required solution, and the vectors representing the terms having been set up in their proper order, the sense of the vector difference is correctly obtained.

The principle applied in the above example need not be confined to absolute and relative velocities of the same inclination, but is universally applicable.

Figure 128 shows a car which carries a wheel, W, supported so that it is free to rotate about center A while the car is either at rest or in motion.

Let the wheel turn about center A with angular velocity, ω, directed clockwise while the car is at rest. Then point B will have linear velocity v_B

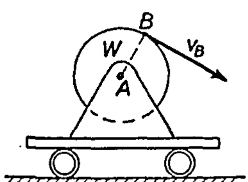


FIG. 128

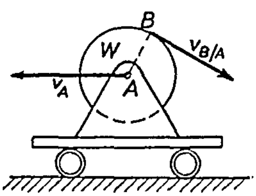


FIG. 129

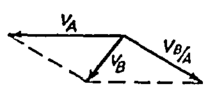


FIG. 130

relative to A equal to $\omega \cdot AB$ and at right angles to radius AB. Since point A is at rest relative to the track, this linear velocity of B is an absolute linear velocity. If the car is now set in motion as represented in Fig. 129 so that point A has an absolute linear velocity v_A, then the previous velocity of B is no longer its absolute velocity, but its velocity relative to a moving point.

To find the absolute velocity of point B, v_B, we must add the absolute velocity of point A, v_A, to the velocity of B relative to A, or $v_B = v_{B/A} \mathbin{+\!\!\!\!\rightarrow} v_A$ (see Fig. 130).

Let us note that the velocity of B relative to A is always perpendicular to AB, that is, tangential to its only possible path of motion about A. We can check this observation by recalling that the points A and B are two points of the same rigid body, and that therefore the distance between them remains unchanged regardless of the motion of the body.

It then follows directly that B can have no component of relative velocity with respect to A in the direction connecting them.

The velocity analysis of plane motion of Art. 48 rested upon segregation of the translation and rotational components whose sum was equivalent to the total plane motion.

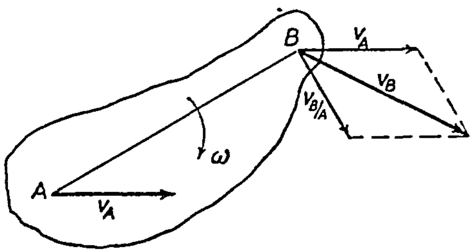


Fig. 131

We may again view plane motion to observe how such a combination of translation and rotation is an application of the principles of absolute and relative velocity.

If, as in Fig. 131, a body is given plane motion, the relationship of the velocities of any two points A and B has been fixed.

If point A is chosen as a reference, the absolute velocity of point B is equal to the sum of the absolute velocity of A plus the velocity of B relative to A, or $v_B = v_A \nrightarrow\!\!\!\!+ v_{B/A}$.

The velocity of B relative to A is

$$v_{B/A} = \omega \cdot r$$

where ω is the angular velocity of the body at the given instant, and r is the distance AB.

This would be the velocity of B relative to A whether the point A were itself in motion or at rest. We have, in this relative motion, a motion of rotation about A as an axis.

We next add to $v_{B/A}$ the absolute velocity of A to obtain the absolute velocity of B. This is equivalent to giving the entire body a motion of translation, imparting to every particle a velocity identical with the absolute velocity of A.

The absolute velocity of point B, the sum of its velocity relative to A plus the absolute velocity of A, is a combination of a translation plus a rotation.

In breaking down this velocity analysis into translation and rotation, points A and B have been taken anywhere on the body, and enjoy no special properties.

We may, therefore, always analyze a plane motion of a body by dividing it into the two elements of translation and rotation. In such a division we may select as a reference any point of the body, and relate other points to it, provided that we know the velocity of the reference point, and the angular velocity of the body.

Frequently we shall find that the known data consist of the absolute linear velocities of two points on the body. It is desired that the angular velocity, ω, of the body be found.

Then we take the difference of the two absolute velocities. This is the relative velocity between the given points, and equals $\omega \cdot r$ where ω is the angular velocity sought, and r is the distance between the two points.

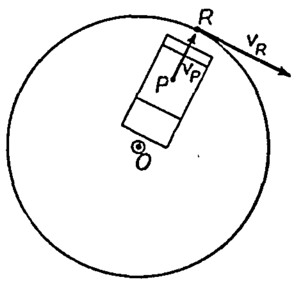


FIG. 132

When two points lie on different bodies, the relative velocity may again be obtained as the vector difference. In this case, we have no clue as to direction of relative velocity, since the two points may now have a component of relative velocity in the direction connecting the two points, which is no longer forced to remain rigid.

In Fig. 132, for example, we find a wheel whose axis O is fixed. Inserted in a slot in the wheel is a small block P. The block is free to move in the guides whether the wheel moves or not, and points on the block may have an absolute velocity in the indicated direction v_p.

Points on the wheel, however, are constrained so that they must move in a circular path about O, and a point R can only have absolute linear velocity in a tangential direction, like v_R. Then points P and R may have entirely different components of velocity in the direction connecting them.

The discussion of absolute and relative velocity has thus far dealt with points, and the velocities we have studied have been linear velocities. The translation element of plane motion is covered by such a discussion, since velocities of translation are equivalent for all points of the moving bodies.

A more adequate background for the study of mechanisms should equip us with the ability to deal with relative velocity properties of bodies in rotation. This requires that we note such properties with lines, rather than points, as our basis of study.

In Fig. 133 we find two rotating bodies, A and B, mounted so that they have a common axis. These two bodies are independent of each other, and either may turn while the other remains at rest, or rotates with the same or different angular velocity as the first.

If lines like ab and cd are marked upon the bodies, we shall be able to

observe the performance of these lines and draw conclusions as to the angular velocity of the rotating bodies.

Line *ab* has the angular velocity of body A, ω_A; and line *cd* has the angular velocity of body B, ω_B.

If we cause both bodies to rotate so that their angular velocities are measured relative to the fixed axis, both ω_A and ω_B will be absolute angular velocities.

The angular velocity of body B relative to A will be the difference be-

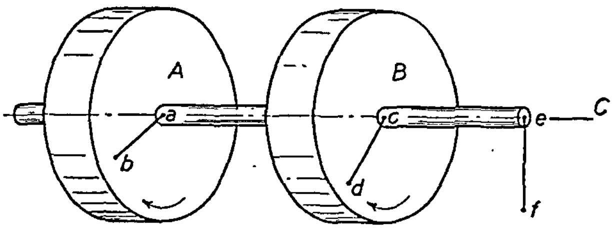


FIG. 133

tween the absolute angular velocity of B and the absolute angular velocity of A, or

$$\omega_B \text{ (relative to } A) = \omega_B \text{ (absolute)} - \omega_A \text{ (absolute), or } \omega_{B/A} = \omega_B \rightarrow \omega_A$$

This is a difference of two vector quantities, and we may subtract vectors to obtain the answer.

Illustrative Example. Line *ab* on wheel A has absolute angular velocity of 5 radians per second, clockwise as we look at the wheel from point C (Fig. 133).

Line *cd*, on wheel B, has absolute angular velocity of 15 radians per second, also clockwise as we look from point C.

Proceeding with vector representation according to the method outlined in Art. 44, we draw vectors ω_A and ω_B as in Fig. 134.

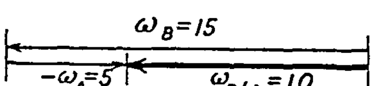


FIG. 134

These vectors represent absolute angular velocities. The angular velocity of B relative to A will be their vector difference, $\omega_{B/A}$, and we note that the angular velocity of B relative to A is 10 radians per second, clockwise as we look toward the cylinders from point C.

The result may be verified by analyzing analytically without the use of vectors.

We consider a line, *ef*, perpendicular to the axis (Fig. 133) to be fixed in space. Line *ab* is rotating clockwise, relative to *ef*, at the rate of 5 radians per second. Line *cd* is going faster, since it is rotating, also clockwise, relative to *ef*, at the rate of 15 radians per second.

If cd is rotating relative to ef at 15 radians per second clockwise, and ab is rotating relative to ef at 5 radians per second clockwise, then cd must be rotating relative to ab at the rate of 10 radians per second, clockwise.

This analysis is quite simply made without recourse to vectors when the two wheels, A and B, have parallel planes of motion. If these planes of rotation are oblique to each other the vector solution will have the advantage, since the subtraction of two vectors is as simple a process when the vectors are oblique as when they are parallel.

PROBLEMS

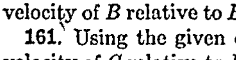
153. A car A is travelling East at the rate of 40 m.p.h. Another car B is travelling West at the rate of 55 m.p.h. Find

(a) the velocity of A relative to B.

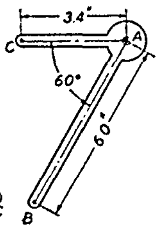
(b) the velocity of B relative to A.

154. A particle, A, travels North with an absolute velocity of 13.0 f.p.s. A second particle, B, travels Southeast with an absolute velocity of 10.8 f.p.s. Find the velocity of B relative to A. *Ans.* $v_{B/A}$ (magnitude) = 22 f.p.s.

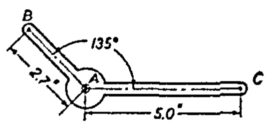
Problems 155–158. The body ABC rotates about fixed axis A with an absolute angular velocity of 1 radian per sec., clockwise. Find the absolute velocities of points B and C, the velocity of C relative to B, and the velocity of B relative to C.



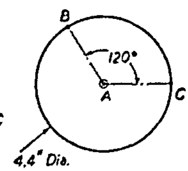
PROB. 155



PROB. 156



PROB. 157



PROB. 158

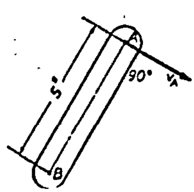
159. Using the given data of Problem 129, find the velocity of E relative to B.

160. Using the given data of Problem 130, find the velocity of A relative to B, the velocity of B relative to E, and the velocity of A relative to E.

161. Using the given data of Problem 133, find the velocity of A relative to C, the velocity of C relative to B. Also determine the angular velocity of the body ABC.

162. Using the data given in Problem 134, find the velocity of C relative to B, and the velocity of B relative to C. Also determine the angular velocity of the body ABC.

163. Point A has an absolute velocity, $v_A = 3.5$ in. per sec. The velocity of point B relative to A is 2.1 in. per sec. in the same sense as v_A. Find the absolute velocity of point B and the angular velocity of body AB. *Ans.* 1.4 in. per sec.; 0.42 radians per sec.



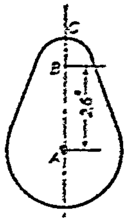
PROB. 163

164. Point C has a velocity relative to point B of 1.6 in. per sec. If the angular velocity of body ABC is 2 radians per sec., find the distance BC and the absolute velocity of point C.

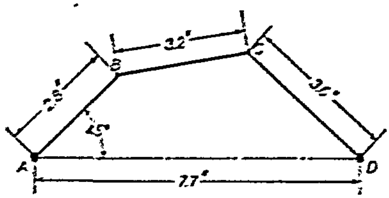
165. The angular velocity of crank AB is 10 radians per min., counter-clockwise. Find the velocity of the instantaneous axis of body BC relative to C.

166. Wheel W has an absolute angular velocity of 300 r.p.m., clockwise. The arm A carrying the axis of the wheel has absolute angular velocity of 500 r.p.m., counter-clockwise.

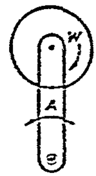
Determine the angular velocity of W relative to A.



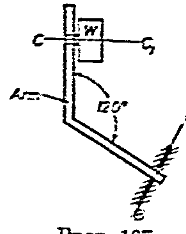
Prob. 164



Prob. 165



Prob. 166



Prob. 167

167. The arm rotates about fixed axis B_1-B carrying with it the axis C_1-C supporting wheel W. If the absolute angular velocity of W is 2 radians per sec. clockwise when viewed from C_1 and the absolute angular velocity of the arm is 3.6 radians per sec. clockwise when viewed from B, determine

(a) *the angular velocity of W relative to the arm.*

(b) the angular velocity of the arm relative to W.

50. Sliding Contact. When bodies are in contact so that one constrains or determines the motion of another, it is possible to define the contact as one of two general types. Figure 135 illustrates *sliding contact.* A block A has been inserted in a slot cut in body B so that the block may slide freely in the slot.

Consider two points which are at the instant in contact, like C of block A, and D of body B.

If, as in Fig. 136, point C is given a velocity Cc' while D remains at rest,

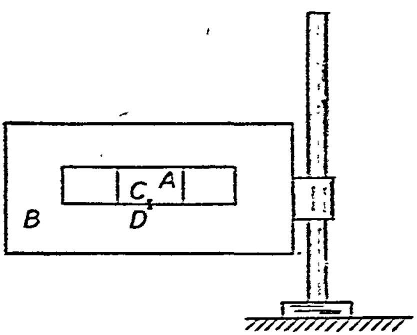


Fig. 135

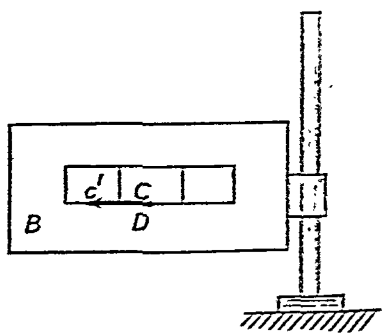


Fig. 136

then Cc' is the velocity of C relative to D, which is also its absolute velocity. The only velocity which C may have relative to D is in a direction parallel

to the sides of the slot which constrain the block, and which we shall call the *sliding surfaces*. This velocity of C relative to D is called the *rate of sliding*, or occasionally "C's slip with reference to D."

If we introduce a new element of motion by allowing body B to move with velocity v_B as in Fig. 137, point D now has velocity, $Dd_3 = v_B$. C has velocity Cc' which may be resolved into two components, one parallel to the sliding surfaces and another perpendicular to the sliding surfaces.

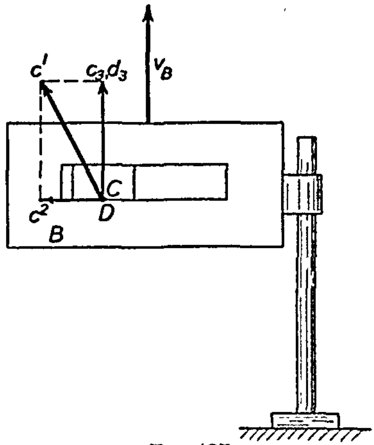


FIG. 137

The component of C's velocity parallel to the sliding surfaces Cc_2 is as independent of the motion of point D as it was when body B was at rest.

The component of C's velocity perpendicular to the sliding surfaces Cc_3 is, however, forced to remain equal to the component of D's velocity in that same direction Dd_3, for C and D may have no relative velocity in a direction perpendicular to the sliding surfaces, since there is no freedom between them in that direction.

Figure 138 shows a mechanism which involves sliding contact.

The large piece M rotates about fixed axis O. A slot is cut in M, and a

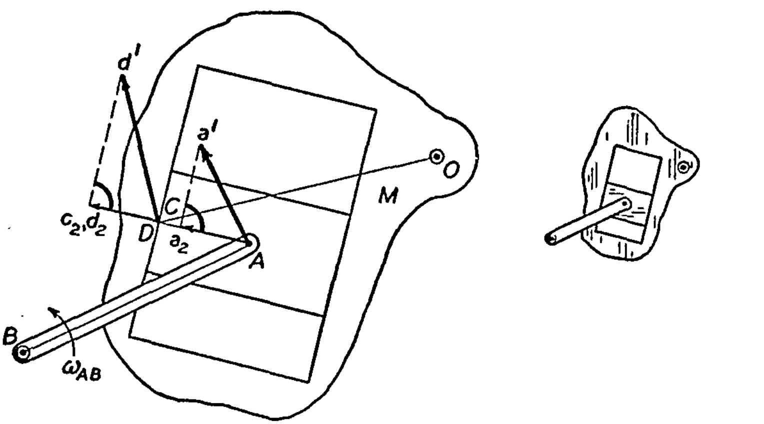


FIG. 138

block inserted which may move freely in the slot. An arm or crank AB is attached to the block by a pin joint at A, and rotates about fixed axis B.

The angular velocity, ω_{AB}, of the arm AB is known, and it is desired that we determine the angular velocity of the piece M.

Aa', the linear velocity of point A on the arm, is readily obtained in direction (perpendicular to AB) and magnitude ($\omega_{AB} \cdot AB$).

The orthogonal component of A's velocity perpendicular to the sliding surface is Aa_2.

Point C, which, like point A, is a point on the small block, must have an orthogonal component in the direction AC which is $Cc_2 = Aa_2$, for these are two points of the same rigid body, and they must therefore have the same orthogonal component in the direction connecting them. Point D on body M must have an orthogonal component perpendicular to the sliding surface which is $Dd_2 = Cc_2$, since there can be no relative velocity between points C and D in this direction.

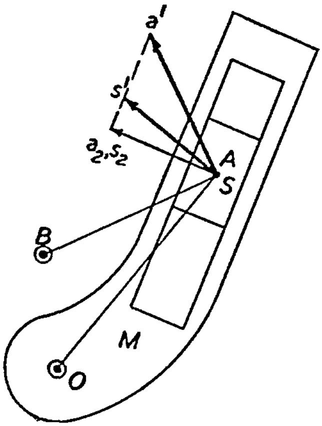


Fig. 139

We now have, for point D, one orthogonal component. Since D is a point on body M, the inclination of its resultant velocity must be perpendicular to DO. Then the resultant velocity of point D becomes known (Theorem I) and is Dd'. If we divide the magnitude of linear velocity Dd' by radius DO, we have the angular velocity of body M.

In most cases of sliding contact, the resolution of velocities is made directly at the pin A of the block as in Fig. 139.

This solution makes use of rigid body properties, as follows: The piece M, from the concept of the rigid body of mechanics, is unlimited in extent. There is then a point S of body M which is located at the same position in space as point A of the block, and such a point may be used to establish velocity relationships. This point of body M must have an orthogonal component perpendicular to the sliding surface, Ss_2 which is equal to Aa_2, and a resultant velocity Ss', perpendicular to SO.

51. The Instantaneous Axis of Sliding Contact. We have already found (Art. 47) that a sliding block constrained to move between fixed, straight-line guides will have an instantaneous axis lying along a line perpendicular to the direction of sliding, and at an infinite distance from the block.

When as in Fig. 140 the guides are placed upon a body which is itself in motion, the instantaneous axis of the sliding block may be found by establishing the direction of the velocity of two points.

In Fig. 140, Aa', the velocity of point A on AB, is given. This is also

the velocity of point A on the block. We may, as in Fig. 139, establish Ss' the velocity of point S on body M, and from it the velocity of any other point, like C, which also lies on body M. Cc' is the resultant velocity of C on body M.

The orthogonal component of C's velocity in a direction perpendicular to the sliding surface at this instant is Cc_2. This is also an orthogonal com-

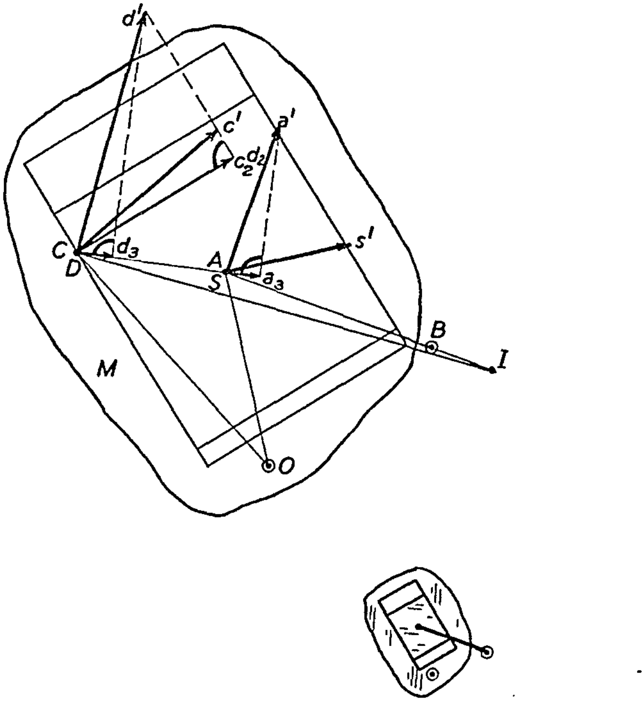


FIG. 140

ponent of the velocity of point D, a point in contact with C, but lying on the block. If now we resolve Aa' into its orthogonal component Aa_3 in the direction AD, and transmit this orthogonal component along the rigid body to point D where it appears as Dd_3, we shall have two orthogonal components of D's velocity. These two satisfy the requirements stated in Theorem II, and Dd' becomes the resultant velocity of point D.

With points A and D possessing known velocities, the instantaneous center of the block is established at I.

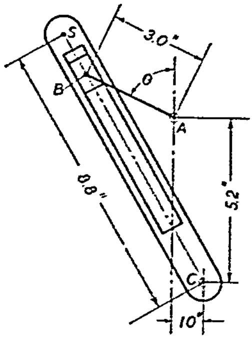
PROBLEMS

168. The angular velocity of crank AB is 2 radians per sec., clockwise. Determine the angular velocity of beam CS when $\theta = 45°$. *Ans.* 0.7 radians per sec.

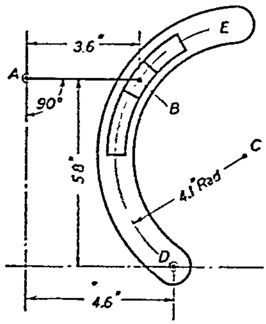
169. Using the dimensions of the oscillating-beam mechanism shown in Problem 168, determine the velocity of point S when $\theta = 60°$, and the angular velocity of AB is 300 r.p.m., counter-clockwise.

170. In Problem 169, find the rate of sliding of point B on the small block relative to point B on the piece SC.

171. Crank AB has an angular velocity of 1 radian per sec. clockwise. Point C is the center of curvature of the slot. Determine the velocity of point C and the angular velocity of body DE. *Ans.* 2.12 in. per sec.; 0.52 radians per sec.

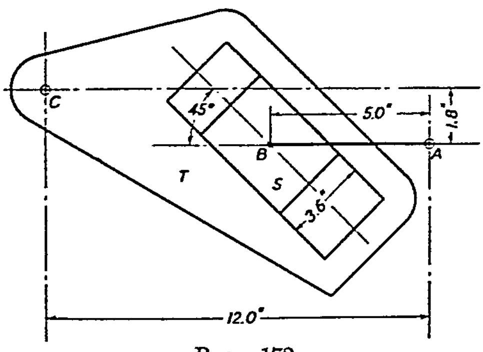


Prob. 168



Prob. 171

172. Crank AB has an angular velocity of 2 radians per sec., clockwise, in the horizontal position shown. The slotted piece, T, rotates about axis C. Determine the instantaneous axis and the angular velocity of block S. Point B is at the center of S.



Prob. 172

173. If the angular velocity of the crank AB, Problem 362, is 1800 r.p.m. clockwise, determine the velocities of pistons C and D, for the position shown. Locate the instantaneous axis of BEC.

174. Crank AB of Problem 363 rotates, clockwise, at 60 r.p.m. When AB makes an angle of 45° with the horizontal, locate the instantaneous axis of EF, and determine the velocity of F.

52. Rolling Contact. When a body rolls upon another, the contact may be such that there is no relative motion between the two points, one lying on each body, which are in contact. This type of motion is called *pure rolling contact.*

To clearly appreciate such a rolling action, let us consider the pair of equal cylinders shown in Fig. 141.

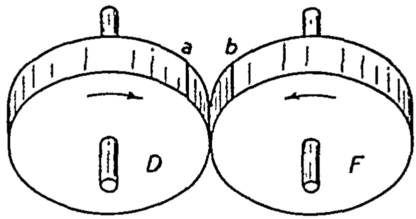


Fig. 141

If both cylinders are rotating with the same angular speed but in opposite directions, lines a and b will become the contacting lines at the same instant, and will then depart allowing other mated lines to come up to contact and depart. Exactly one revolution after the time that a and b are in contact they will again meet, and will continue to do so as long as the wheels turn.

At all times lines a and b have the same speed, and at the instant of contact, since their direction of motion is the same, they will have the same velocity. There will then be no relative velocity between the contacting lines, and the bodies are in pure rolling contact.

If we now interrupt this smooth, regular rolling action by holding the follower still while the driver rotates, line a of the driver will have an absolute velocity while it passes the contacting position, but the contacting line of the follower, being at rest, will have zero absolute velocity. There is now relative velocity between the contacting lines of the two bodies, and these lines are sliding or "slipping" by each other. In this case, the slipping is complete, and the relative velocity between the contact lines is equal to the entire absolute velocity of the moving line.

If the driver is now given an angular velocity which is different than the angular velocity of the equal-size follower, the two lines which come into contact will have different velocities at the instant of contact. The difference of the velocities—the relative velocity—will be the rate of sliding which will not be a complete sliding, as in the previous illustration, but a fractional or partial slip.

The velocity analysis of such slip is a case to be explored with the methods outlined in Art. 50 for any sliding contact. The contacting bodies need not be equal cylinders; whenever two bodies are in contact so that there is no relative velocity between contacting lines or points, they are in pure rolling contact.

53. Instantaneous Axis of Rolling Contact. When a wheel, W, has a motion of pure rolling contact with its track, as in Fig. 142, the points of

contact A on W_1 and B on the track must have the same velocity. If the track is at rest, as when it is fixed to the earth, point B has no velocity. Point A then becomes a point of zero velocity on body W_1 and hence its instantaneous center.

If the track or body on which W_1 rolls is itself in motion, as in Fig. 143, and the motion is one of pure rolling contact, the instantaneous center of W_1 is no longer at point A, which now has linear velocity.

Wheel W_2 is turning about fixed axis O with angular velocity ω_2. Then point B of body W_2 has linear velocity $Bb' = \omega_2 \cdot OB$. Point A has linear velocity $Aa' = Bb'$ since there is no slip.

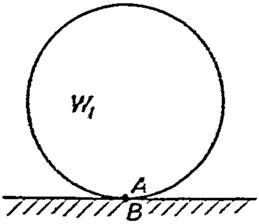


Fig. 142

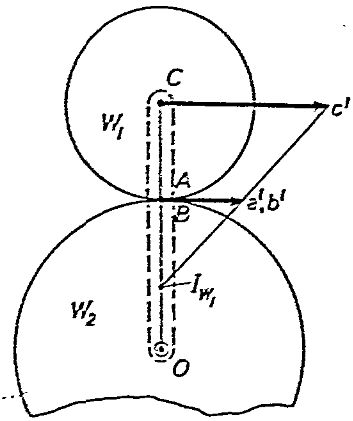


Fig. 143

The arm which carries the axis of W_1, point C, is turning about fixed axis O with angular velocity ω_A. Then point C has linear velocity $Cc' = \omega_A \cdot OC$.

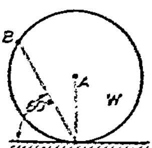
Since the velocities of two points, A and C on body W_1, are known, the instantaneous axis may be found, and is determined as I_{W_1}.

PROBLEMS

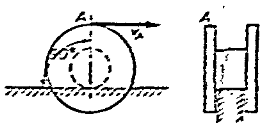
175. Cylinder W, 4 in. diameter, is in pure rolling contact with a fixed track. The angular velocity of the cylinder is 4 radians per sec. clockwise. Find the velocity of point A, the center of the cylinder, and of point B on its surface.

Ans. 8 in. per sec.; 13.9 in. per sec.

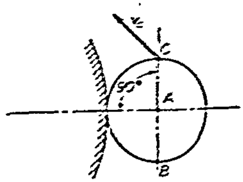
176. The drum, consisting of a central shaft and discs fastened to the shaft, rolls without slip on the fixed track. The diameter of the shaft is 1.80 in. and the diameter of



Prob. 175



Prob. 176



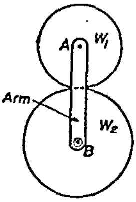
Prob. 177

the discs is 2.72 in. Given the velocity of point A, $v_A = 4$ in. per sec., determine the angular velocity of the drum and the velocity of the top of the shaft.

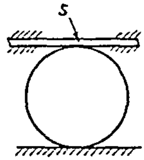
177. The disc, 6 in. diameter, rolls without slip on the curved fixed track. Given the velocity of point C, $v_C = 2$ f.p.m., determine the velocities of points A and B.

178. Wheel W_1, 4 in. diameter, is supported on axis A carried by the arm, and is in pure rolling contact with W_2, 5.3 in. diameter. The arm and W_2 are mounted upon fixed axis B. *The angular velocity of the arm is 2 radians per sec. clockwise, and the angular* velocity of W_2 is 1 radian per sec. clockwise.

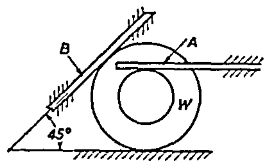
(a) Locate the instantaneous axis of W_1 and determine its absolute angular velocity.
(b) Determine the angular velocity of W_1 relative to the arm.



PROB. 178



PROB. 179



PROB. 180

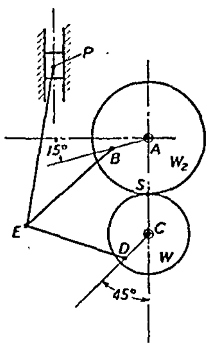
179. The wheel is in pure rolling contact with the fixed track and with the bar S, which slides parallel to the track. The diameter of the wheel is 4 in., and its angular velocity is 150 r.p.m., counter-clockwise. Find the velocity of S. *Ans.* 314 f.p.m.

180. Wheel W is in pure rolling contact with sliding bars A and B. A moves parallel to the track, and B moves at an angle of 45° with the track. If the velocity of A is 2.5 in. per sec. to the left, find the velocity of B, the angular velocity of W and the rate of sliding of W on the fixed horizontal track. The diameter of the wheel is 5.3 in. and the diameter of the concentric projection which is in contact with A is 2.7 in.

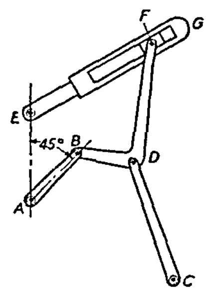
181. Wheel W of Problem 316 has an angular velocity of 120 r.p.m., clockwise, in the position shown. Determine the angular velocity of AB, and locate the instantaneous axis of BC.

182. Crank AB of Problem 317 has an angular velocity of 1 radian per sec., clockwise. Locate the instantaneous axis of W_2 and determine: (a) its absolute angular velocity; (b) the angular velocity of W_2 relative to W_1.

183. The mechanism of a differential stroke engine is shown. Wheels W_1 and W_2 are in pure rolling contact at S. $AB = 1.9$ in.; $AS = 2.7$ in.; $SC = 2.0$ in.; $CD = 1.7$ in.; $DE = 5.0$ in.; $BE = 5.7$ in.; $PE = 7.7$ in. The center line of P is 4.5 in. to the left of A. If W_1 has an angular velocity of 1 radian per sec., clockwise, determine the linear velocity of the piston P.



PROB. 183



PROB. 185

184. If the piston, P, of Problem 183 has a linear velocity of 20 in. per sec., in the position shown, determine the angular velocities of wheels W_1 and W_2.

185. The mechanism is driven by crank AB. A, C, and E are fixed axes. E is 4.2 in. above A. C is 3.9 in. below and 7.0 in. to the right of A. $AB = 3.3$ in.; $BD = 2.7$ in.; $CD = 6.0$ in.; $DF = 6.0$ in. Angle BDF of the rocker arm $= 90°$. The angular velocity of $AB = 600$ r.p.m., counter-clockwise. For the position shown,

(a) Locate the instantaneous axis of the rocker arm.
(b) Determine the angular velocity of EG.
(c) Determine the rate of sliding of the block at F relative to the slot.

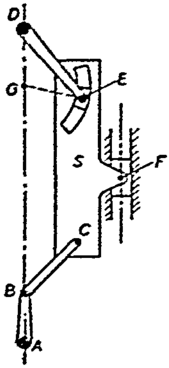
186. Cranks AB and DE are geared together (gearing not shown) so that AB makes 10 revolutions while DE makes 3 in the opposite direction. The block at E slides in a slot, cut in piece S, with center of curvature at G. The block at F slides in vertical fixed guides, and is pinned to S. $AD = 10.0$ in.; $AB = 1.4$ in.; $BC = 2.30$ in.; $DE = 2.8$ in.; $GE = 1.8$ in. Rad.; $CF = 2.4$ in.

The center line of the block at F is parallel to AD, and 3.0 in. to the right of it.

In the position shown, AB is on AD, $DG = 1.8$ in.; $EC = 4.5$ in.

If the angular velocity of AB is 60 r.p.m., clockwise,

(a) Locate the instantaneous axis of S.
(b) Determine the linear velocity of F.
(c) Determine the absolute angular velocity of S.
(d) Determine the rate of sliding of the block at E in the curved slot.



Prob. 186

54. Friction Wheels. Rolling Bodies. We have already noted, in our preliminary survey of mechanisms (Art. 17), the class known as rolling-contact bodies or friction wheels. Let us now examine these bodies in greater detail, adding a study of their property of velocity to our kinematic information. In all cases of rolling contact, it will be assumed that provision has been made for sufficient frictional resistance to insure no slipping, and we shall concern ourselves with the resulting kinematical action.

55. Rolling Cylinders. The pair of cylinders shown in Fig. 144 is in pure rolling contact.

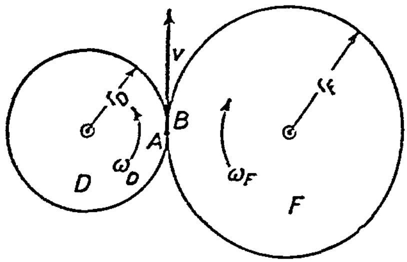


Fig. 144

Then if point A on driver D, and B on follower F are in contact, they have the same velocity, $v = v_A = v_B$.

But $\quad v_A = \omega_D \cdot r_D$

where $\quad \omega_D$ is the angular speed of the driver
and $\quad r_D$ is the radius of the driver.

Also
$$v_B = \omega_F \cdot r_F$$

where $\quad \omega_F$ is the angular speed of the follower
and $\quad r_F$ is the radius of the follower.

Then
$$\frac{v_A}{v_B} = 1 = \frac{\omega_D \cdot r_D}{\omega_F \cdot r_F}$$

and
$$\frac{\omega_F}{\omega_D} = \frac{r_D}{r_F}.$$

Or the angular speeds of the follower and driver are in inverse ratio to their radii.

Or, since

$$\frac{\omega_F}{\omega_D} = \frac{2r_D}{2r_F} = \frac{D_D}{D_F}$$

where D_D is the diameter of the driver,
and D_F is the diameter of the follower,

the angular speeds of driver and follower are in inverse ratio to their diameters. The ratio $\frac{\omega_F}{\omega_D}$ is called the *speed ratio*, and, in this text, the angular speed of the follower will always be made the numerator of the speed ratio.

When the contact is *external*, the angular velocities are of opposite direction. When the contact is *internal*, the angular speed ratio is again fixed by the two diameters but, in this case, the velocities are of the same direction.

In the rolling-cylinder drive of Fig. 144, the analysis of the angular velocities of contacting bodies has centered in the linear velocity of the contacting points A and B. In all velocity studies of contact between two bodies, the velocity of the point of contact is the connecting medium in transmitting the motion, and, in our analyses, we should always seek to establish that velocity as the connecting medium in the relationships we derive in a search for speed ratios, and directions of velocities.

PROBLEMS

187. The distance between the centers of two rolling cylinders in contact is 18 in. If the speed ratio is 3 : 1, and the cylinders rotate in opposite directions, calculate their diameters. *Ans.* $D_D = 27$ in.; $D_F = 9$ in.

188. Two rolling cylinders are in contact and have the same direction of rotation. Calculate their diameters if the distance between centers is 5.20 in. and the speed ratio is 2 : 1.

189. Solve Problem 188, with the speed ratio changed to 3.5 : 1.

56. Rolling Cones. When the shafts which are to be connected by rolling bodies are not parallel, solid bodies other than cylinders must be employed.

In the case of intersecting shafts which are oblique to each other, the solids become cones, whose elements are the contacting lines. These lines must have the same absolute velocity at the instant of contact if a pure rolling action is to be obtained.

We shall examine an illustration to develop an expression for the speed ratio of these rolling bodies.

In Fig. 145, two cones whose shafts or axes make an angle θ with each other are in contact.

There is pure rolling contact, and points c and l must have the same

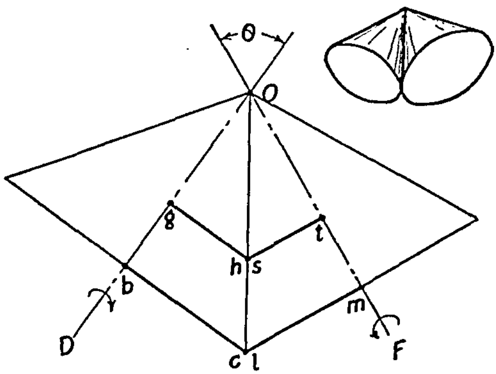


FIG. 145

velocity, as must all contacting points which lie along the contacting elements.

Then

$$v_l = \omega_F \cdot lm$$

and

$$v_c = \omega_D \cdot bc$$

Now dividing,

$$\frac{v_l}{v_c} = \frac{1}{1} = \frac{\omega_F \cdot lm}{\omega_D \cdot bc}$$

or

$$\frac{\omega_F}{\omega_D} = \frac{bc}{lm}.$$

But bc is the radius of the driver at the point of contact,

and lm is the radius of the follower at the same point.

Then the speed ratio of two rolling cones is equal to the ratio of the radius of the driver to the radius of the follower, both radii being measured at the same point of contact.

If any other pair of radii from a point of contact be selected, it will be noted that the speed ratio is not affected. For example, when we consider contacting points h and s, the ratio of radii is $\frac{gh}{st} = \frac{bc}{lm}$ because they are corresponding sides of similar triangles.

Having established the speed ratio, we should, to complete the velocity description, consider the directions of velocity.

It will be necessary to adopt a conventional system for describing direction. Therefore we will agree that we fix direction by looking from the large diameter side of a cone toward the apex of intersection in every case.

In Fig. 145 if direction is established in that convention, we find that the driver is turning clockwise, and the follower counter-clockwise. Then the cones have opposite direction of velocity, and this will always be true when the cones are, as shown, in external contact.

The analytical expression for speed ratio may be derived as follows:

$$\frac{\omega_F}{\omega_D} = \frac{bc}{lm} = \frac{Oc}{Oc} \cdot \frac{\sin bOc}{\sin lOm} = \frac{\sin bOc}{\sin lOm}$$

$$= \frac{\sin(\theta - lOm)}{\sin lOm} = \frac{\sin\theta \cos lOm - \cos\theta \sin lOm}{\sin lOm}$$

$$= \sin\theta \cot lOm - \cos\theta.$$

Since both θ and desired speed ratio are available when selecting a pair of rolling cones, the analytical expression will make it possible to calculate the cone angles of both cones.

For example, we are given the problem of designing two cones so that the speed ratio will be 3 : 1. The shafts of driver and follower intersect at 42°.

Then

$$\frac{\omega_F}{\omega_D} = \frac{3}{1} = \sin 42^\circ \cot lOm - \cos 42^\circ$$

$$3 = 0.6691 \cot lOm - 0.7431$$

$$\cot lOm = \frac{3.7431}{0.6691} = 5.594$$

$$lOm = 10^\circ\, 8'.$$

Cone angle of follower = 20°16′.

Cone angle of driver = 63°44′.

In actual use the entire cone is unnecessary, and frustums of cones are employed.

The design of a pair of rolling cones may be simplified by turning to a graphical solution. Let us illustrate by designing a pair of rolling cones to accomplish the drive of the previous example.

We are again given two intersecting shafts which make an angle of 42°

with each other. These shafts are to be connected by rolling cone frustums $\frac{3}{4}$ in. deep so that the speed ratio will be 3 : 1, and the shafts are to have opposite directions of rotation.

Figure 146 shows the graphical method of solution. Starting with axes of D and F 42° apart, we draw a line ab parallel to the driver's axis at a distance of 3 units from it, and another line ef parallel to the axis of the follower at a distance of one of these same units from it. The intersection S of the parallels to the axes of driver and follower determines the contacting element OS.

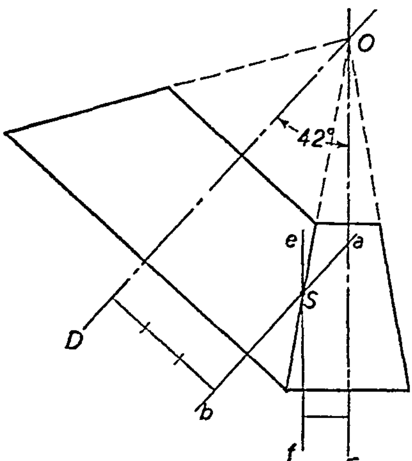


Fig. 146

At any desired point along the axes the frustums of $\frac{3}{4}$ inch depth are drawn as indicated.

Quite as necessary as the solution which we have now accomplished is the check-up. Analyzing these frustums for speed ratio we note that since any radius of the follower, as that at S, is $\frac{1}{3}$ as large as the corresponding radius of the driver, then $\frac{\omega_F}{\omega_D} = \frac{3}{1}$, which is the speed ratio sought. Since the cone frustums are in external contact, the direction of rotation is opposite. It is always good practice to pause to check the solution to note whether or not it has fulfilled the original demands.

Like cylinders—and rolling cones are, in effect, a series of successive laminated planes which are individually circles—an internal contact arrangement will give the same direction of rotation to both cones. Figure 147 illustrates a case in which a smaller cone has been mounted inside a large hollow cone. It will be noted that with pure rolling contact (contact point on both cones has same velocity) the follower will have the same direction of rotation as the driver. The design procedure is identical with that employed for the external contact type.

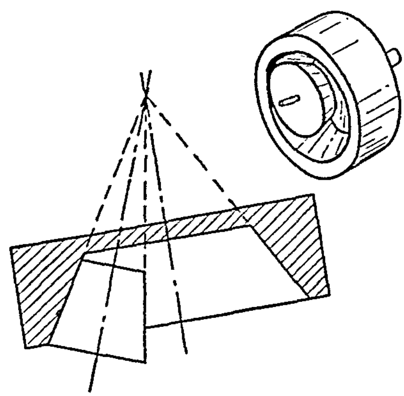


Fig. 147

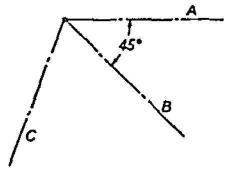
PROBLEMS

190. The angle between two intersecting shafts is 37°. Design two rolling cones to have a speed ratio of 2.8 : 1, and opposite directions of rotation. Use a graphical solution, checking by calculating the cone angles.

191. Two rolling cones are placed in external contact with shaft angle equal to 60°. If the driver is to make 60 r.p.m. while the follower makes 25 r.p.m., design the cones.

192. Design two rolling cones, to be in internal contact with shaft angle 40° and speed ratio of 3.2 : 1. Base of smaller cone = 1 in.

193. Three shafts are to be connected by rolling cones in external contact. The speed ratio of $A : B : C$ is 5 : 2 : 1.5. Base diameter of cone on shaft B = 2 in. Design the cones, graphically, and report cone angles and base diameters.



PROB. 193

57. Disc and Wheel. The cone angles of the rolling cones which have been discussed may be any angle. If one cone has a cone angle of 180°, as shown in Fig. 148, that cone becomes a flat disc. We find this form, in

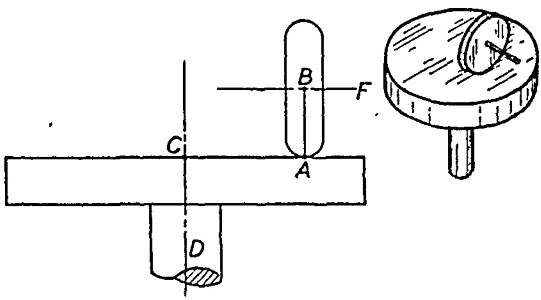


FIG. 148

practice, further modified by reducing the depth of the frustum of the small cone. If the depth becomes infinitely smaller, that is, becomes a single plane, the smaller cone, now a wheel of one-plane thickness, may be mounted upon an axis which is at right angles to the shaft of the disc.

In this disc-and-wheel mechanism we have the basis of a change-speed device. The smaller wheel cannot, of course, in practice be made a single plane in thickness, so that except at the point A there is a small amount of slipping and the wheel must be surfaced with a yielding material. Kinematically, we consider the small wheel to be of radius AB. In the position shown, with pure rolling contact

$$\omega_F = \frac{v_A}{AB}$$

and

$$\omega_D = \frac{v_A}{AC}$$

then

$$\frac{\omega_F}{\omega_D} = \frac{AC}{AB}.$$

Now AB is a fixed distance, but the radius AC may be changed at will by moving the wheel across the surface of the disc. As the wheel approaches point C from its present position we observe that the speed ratio AC/AB is becoming a smaller quantity, and if the driver turns at constant speed the follower's speed is diminishing. Thus by setting the small wheel at the correct position, we may operate the follower shaft at any desired speed within the radial-distance limits. When the wheel is at the center of the disc its speed will be zero. Continuing to the left, the direction of rotation of the wheel is now reversed; the disc and wheel transmission therefore offers several speed ratios forward, a neutral position, and several speed ratios backward. It has been used in such drives as automotive transmissions, but unfortunately fails as a successful transmission when called upon to furnish positive action with large amounts of power to be transmitted, in common with the other forms of frictional resistance bodies.

A modification of this application of rolling contact is illustrated in Fig. 149, and a previous example was offered in Fig. 40.

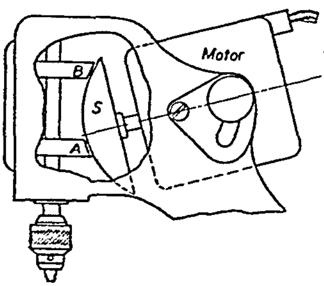
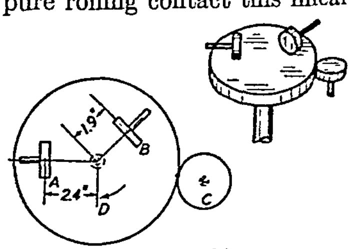


Fig. 149

Figure 149 illustrates a tapping attachment. The motor may be tilted so that the spindle driving the chuck may be given different speeds through the rolling contact. If the cutting stroke drive is from S through A, in the position shown, the tap will revolve at a slower speed than that of the return or tap-withdrawing stroke, S through B. Within the limits of available radii, the speed ratio is infinitely variable, as well as quick-return.

In every case of rolling contact the linear velocity at the point of contact should be established on the driver from the known angular velocity and radius to the point of contact. With pure rolling contact this linear velocity is transmitted to the follower, whose angular velocity is therefore determined.



Prob. 194

PROBLEMS

194. The small rollers A, B, and C are in pure rolling contact with the driving disc, D. The diameter of A is 1.80 in., that of B = 1.65 in., and that of C = 2.37 in. The diameter of disc D is 7.8 in. If the angular velocity of D is 220 r.p.m., clockwise, determine the angular velocities of A, B, and C.

195. The wheel of the disc-and-wheel mechanism shown in Fig. 148 may be moved 5 in. to the right or left of the neutral position at C. The speed of the disc is 100 r.p.m. Plot a curve showing the relationship between angular velocity of wheel F and its position (distance CA) for the complete travel across the disc. Diameter of wheel = 1.5 in.

58. Non-Circular Rolling Bodies. As long as the contacting elements of bodies, whether they be points or lines, have the same velocity they roll without slip. The outlines of the driving bodies may be of great variety, and mating bodies may be designed to roll with them. One example of non-circular rolling bodies is the pair of equal ellipses which are mounted, as in

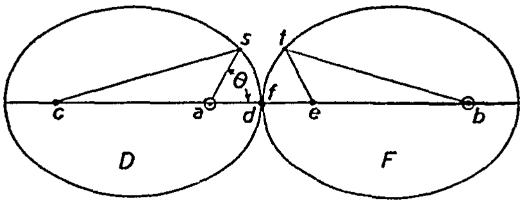


FIG. 150

Fig. 150, upon fixed axes a and b at foci of the ellipses, so that the distance ab = major axis of either ellipse.

At the instant shown, point d of the driver D is in contact with point f of follower F, and the contact is on the line joining centers a and b. At all instantaneous positions of contact the contact must take place on the line of centers ab, for only when they lie on this line can contacting points have identical directions of velocity, which is one of the two necessary conditions for pure rolling contact, the other being equal magnitudes of velocities.

If we draw lines, like sa and sc of Fig. 150, from any point s on the outline of an ellipse to the foci, the sum of the two distances $sa + sc$ is equal to the major axis of the ellipse. (Let us also bear in mind that we are dealing with equal ellipses, and that the distance ab is not only fixed, but equal to the major axis.)

On the follower a point t has been located so that arc ft = arc ds. Then since these are equal ellipses radius tb will equal radius sc and radius te will equal radius sa.

Now let us start the action by turning the driver clockwise through angle θ. Point s now lies on the line of centers ab, as in Fig. 151. If point s is remaining in rolling contact with the other body point t also will have reached the line of centers, for sa must be mated with a radius of length bt, in order that the distance between a and b remain constant and bt is the mating radius which is equal to cs. During this action there has been no slipping for equal arcs ds and ft have been passing the contact point in the

same time interval, and s and t must have identical velocities, of the same direction (perpendicular to ab) and of equal magnitude.

We have rested on the assumption that s became a contact point when it reached the line of centers ab. Assumptions must be checked. We note that if, when s reaches the line of centers, contact were elsewhere along the ellipse surfaces, for example at g, then focus b must have moved, as shown by the dotted outline. When two curves touch they must have a common tangent. In the case of an ellipse this tangent must make equal angles with the radii to the foci. Then angle β_1 must always equal angle β_2. If contact could take place at any other point than s, for example g, the fol-

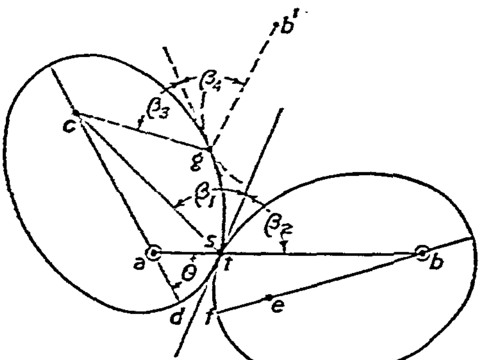


Fig. 151

lower ellipse must be so located that the common tangent to the ellipses will make equal angles β_3 and β_4 with the radii gc and gb'. But our given conditions fix point b at one location.

We conclude that with b remaining fixed, all contact between the ellipses takes place on the line of centers ab, and the assumption from which we drew the original conclusion of pure rolling contact has been checked.

The speed ratio of the rolling ellipses may now be established. Since the contacting points s and t of Fig. 151 have identical velocity, v, then

$$v = \omega_D \cdot as = \omega_F \cdot bt$$

or

$$\frac{\omega_F}{\omega_D} = \frac{as}{bt}.$$

If the driver rotates at constant angular velocity the follower will, during a large portion of its complete cycle, be rotating with slowly changing angu-

lar velocity, and during the remainder, or smaller portion, will have a rapidly changing angular velocity. The actual trend of the speed ratio has been made the basis of one of the problems of the accompanying set.

It should be noted that a pair of rolling ellipses cannot produce a positive motion drive through an entire revolution. Assisting devices, like the cutting of gear teeth on the surfaces, or the provision of link-work must be employed. These are discussed in later articles dealing with those subjects.

Cylindrical, conical, and elliptical rolling bodies have already been discussed. At this point we should establish a method of attack upon rolling bodies of any shape, so that a general method may be available.

Let us assume that the driver, D, of Fig. 152 is given, and it is desired that we design the outline of a follower, to be mounted upon a fixed axis, b, at a known distance from a, the fixed axis of the driver. Pure rolling contact is to be maintained between the bodies.

Since at all instantaneous positions of the contact the contacting points must have the same direction of velocity, they must all lie on line ab.

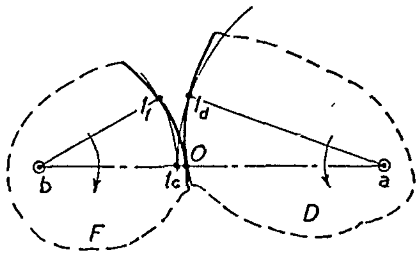


FIG. 152

Then any point 1_d of the driver may be selected and radius $a1_d$ swung to intersect ab at 1_c, which will be the point of contact between 1_d of the driver and 1_f of the follower. To insure this mating we swing radius $b1_c$ about center b and, measuring chord $O1_d$, lay it out from point O to intersect the arc 1_c1_f. The intersection is point 1_f, one point on the follower's surface. A large number of points may be established by repetition of this procedure to obtain the follower's outline, which is their locus.

We note by way of check that during the time the driver is turning to bring 1_d over to position 1_c the driver's outline will be rolling upon the outline of the follower. At the instant 1_d is at 1_c, 1_f will also be there; contact is established, and both points have the same direction of velocity. Their magnitudes will be approximately equal, since chordal distances $O1_d$ and $O1_f$, not arc distances, have been equalized. Such approximations may be refined by selecting smaller chordal distances, but it becomes evident that only when we have selected an infinite number of points to establish the follower's outline will we have abandoned approximation and have arrived at an exact solution. Now, equally evident is the impossibility of selecting an infinite number of points. As in any plotted curve we proceed with caution, selecting fewer points when the curvature is gradual and many when it is rapid or abruptly changing—fairing in a curve or locus from these points. The approximation is not only necessary but justified

by the magnitudes of the deviations, which are slight when the drafting is accurately performed.

It will be noted that when the driver of Fig. 152 has rotated through any angle, the follower will not have turned through the same angle.

It is, therefore, unreasonable to assume that when the driver has made one complete revolution, the follower will have completed one revolution. Then, if we start with one of a pair of rolling bodies as driver, we cannot

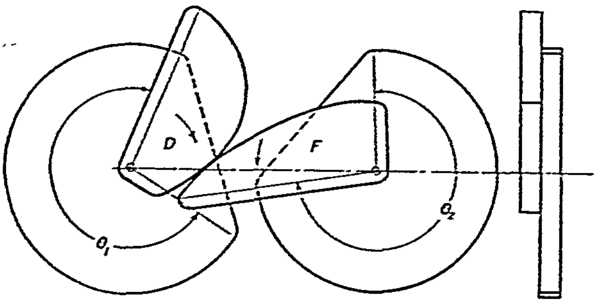


Fig. 153

expect that a continuous and closed curve will result on the follower unless the curves are, as we have already noted, cylinders or ellipses. In the case of most other curves, the mated curve which we design on the follower is either not a continuous curve or not a closed one.

To have a satisfactory drive, that is, one in which both rolling bodies will return to the starting-point together, we can resort to such devices as that shown in Fig. 153.

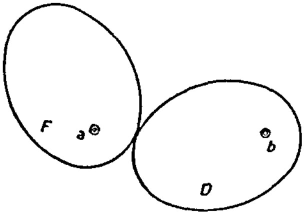
Sectors of two rolling bodies have been designed, as in the previous case. For the balance of the total revolution we may mount rolling cylinders upon the same shafts as D and F, but in a different parallel plane than that of the rolling sectors. If the radii of the rolling cylinders are made inversely proportional to the angles θ_1 and θ_2, the driver and follower shafts will return to the starting position at the same time.

In many mechanisms, the speed ratio during only a portion of the stroke is intended to be variable. In such applications the device of using rolling cylinders to complete a continuous motion is a satisfactory solution.

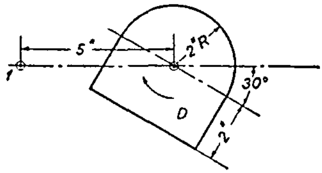
PROBLEMS

196. The *equal* ellipses shown are in pure rolling contact. The angular velocity of D is 1 radian per sec., clockwise. Plot a curve showing the speed ratio of F to D for a complete revolution. Major axis = 4 in. Fixed foci at points a and b, with ab = 4 in. Foci of each ellipse are 2 in. apart.

197. Plate D is to oscillate through an angle of 90°, clockwise, from its present position. A plate F mounted upon axis f is to be driven by D with pure rolling contact. Design the plate F, and plot a curve giving the speed ratio against the angular displacement of D.



PROB. 196

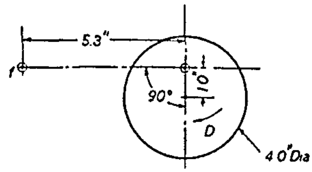


PROB. 197

198. The eccentric plate D oscillates through an angle of 120°, clockwise, from its present position.

Design plate F, which is to be mounted on axis f, and to be driven by D in pure rolling contact. Report the speed ratio, $\frac{\omega_F}{\omega_D}$ when D has turned 30°, 55°, 75°, clockwise, from its present position.

Through what angle does F turn while D turns 120°?



PROB. 198

199. Two axes, d and f, are 6 in. apart. Non-circular plates D and F are mounted upon axes d and f, respectively, and roll on each other with no slip. The speed ratio, $\frac{\omega_F}{\omega_D}$, is given in the table, in terms of the angular displacement, θ, of D. D rotates clockwise.

(a) Design the plates.

(b) Measure the angular displacement of F for the 180° rotation of D, and check the result by plotting and graphically integrating the angular velocity-time curve of plate F.

θ_D —	0°	30°	60°	90°	120°	150°	180°
$\frac{\omega_F}{\omega_D}$ —	0.78	0.90	1.16	1.53	1.77	1.94	2.00

59. Series of Rolling Bodies. The single pairs of rolling bodies may be combined, as in Fig. 154, into a series or train of bodies. The speed ratio of

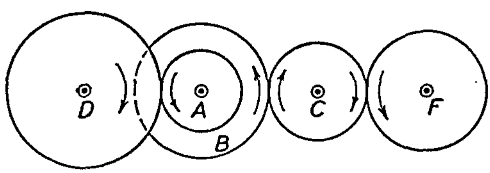


FIG. 154

a train, that is, the ratio of the angular velocity of the final body, or follower, to the angular velocity of the initial body, or driver, $\frac{\omega_F}{\omega_D}$, is called the *train value*, and will be conventionally symbolized here as **t.v.** Then

$$\text{t.v.} = \frac{\omega_F}{\omega_D}.$$

When the shafts connected by a train of wheels are parallel, the train value is called positive (+) if both shafts have the same sense of angular velocity, and negative (−) if they have opposite sense.

In deriving the train value of a series like the rolling cylinders of Fig. 154 we note that at all points of contact the action is pure rolling, and the pairs of contacting points have equal velocity.

In the first pair of bodies, D and A, D is a driver and A is a follower. Then

$$\frac{\omega_A}{\omega_D} = \frac{D_D}{D_A}$$

where D_D is the diameter of the driver,

and D_A is the diameter of the follower.

But A and B are mounted upon the same axis and fixed to each other, so that $\omega_A = \omega_B$.

Proceeding to the next pair of rolling cylinders B and C, with B acting as driver and C as follower,

$$\frac{\omega_C}{\omega_B} = \frac{D_B}{D_C}.$$

In the next pair, we find C now acting as driver and F as follower. Then

$$\frac{\omega_F}{\omega_C} = \frac{D_C}{D_F}.$$

Now the terms of the train value may be assembled by multiplying, as follows:

$$\frac{\omega_A}{\omega_D} \times \frac{\omega_C}{\omega_B = \omega_A} \times \frac{\omega_F}{\omega_C} = \frac{\omega_F}{\omega_D} = \text{t.v.}$$

and

$$\text{t.v.} = \frac{D_D}{D_A} \times \frac{D_B}{D_C} \times \frac{D_C}{D_F}.$$

The term D_C will cancel, since a wheel acting as both driver and follower will present its diameter in both the numerator and denominator of the ratio. Such a wheel is called an *idler*, and its purpose is to reverse the direction of the follower. We may summarize the train value by stating that the train value of a series of rolling cylinders, $\frac{\omega_F}{\omega_D}$, is *equal to the product of the diameters of all drivers divided by the product of the diameters of all followers.* The directional relationship of the follower to the driver of the train is de-

termined by drawing arrows indicating direction on each wheel, as in Fig. 154. We find that if the driver is turning clockwise, then wheel A in rolling contact with it will rotate counter-clockwise, as will B, which is attached to it. C will then rotate clockwise, and F will rotate counter-clockwise.

Then the follower is turning in the opposite direction from the driver.

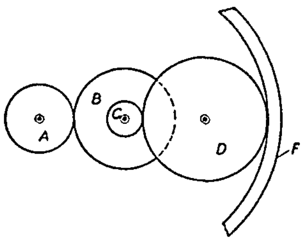
"Chasing through" the series with arrow-heads is necessary to fix the direction of drive, because no formula for fixing direction based upon the number of bodies is satisfactory—for one pair of wheels may yield the same or opposite directions of drive depending upon their arrangement in either internal or external contact.

PROBLEMS

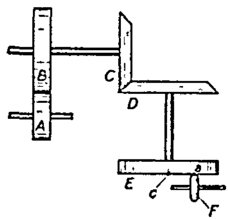
200. The cylinders of a machine-tool drive are in pure rolling contact. Diameter of $A = 5.5$ in.; $B = 8.5$ in.; $C = 3.0$ in.; $D = 10.0$ in.; $F = 23.0$ in.

B and C are fastened together, and the axis of F is coincident with their axis.

Determine the train value if A is the driver, and F is the follower. *Ans.* +0.084



PROB. 200



PROB. 202

201. If the angular velocity of B (Problem 200) is 100 r.p.m., counter-clockwise, determine the angular velocities of A and of F, respectively.

202. The series of rolling bodies is a feed mechanism of an automatic machine. All shafts are mounted in fixed bearings.

Diameter of $A = 4$ in.; $B = 7$ in.

Speed ratio of cones C and $D = 3:4$.

Diameter of $F = 1$ in.

Determine the train value when roller F is placed with a 2 in. to the right of the neutral position, c. A is the driver.

203. Determine the train value, Problem 202, if a is placed 3.85 in. to the left of c. *Ans.* +3.3

60. Flexible Connectors. The kinematic features of flexible connectors and friction wheels are closely related. This group comprises belt and chain drives, which are grouped as flexible connectors because their form changes while in motion. The belt drive is dependent, as are friction wheels, upon friction alone, while chain drives, like the toothed wheels of gearing, form a positive-motion connector.

61. Belt Drives. As in the case of friction wheels, belt drives rely upon frictional resistance to insure a no-slip condition between belt and pulley. Beyond the limit imposed by frictional resistance there is freedom to slip. This condition is not always a disadvantage; in the case of machinery which is continually being started, stopped, or reversed, slipping of the belt will diminish or relieve the tendency of suddenly applied loads to cause shock.

Belts are elastic and they have a finite thickness. When we consider the belt and pulleys of Fig. 155 where the belt has no thickness, but is assumed

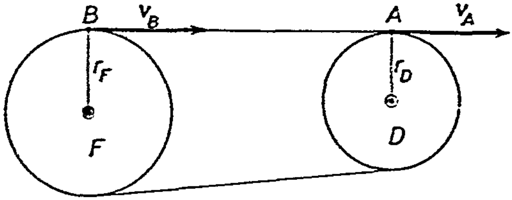


Fig. 155

to consist of a single line, we note that a point A of the driver will have velocity $v_A = \omega_D \cdot r_D$. If there is no slip here, v_A will become the velocity of the contacting point on the belt and will be transmitted along the belt to point B, where if there is again no slip, the velocity of point B on the follower will be $v_B = v_A$

and, since

$$v_B = \omega_F \cdot r_F$$

then

$$\omega_D \cdot r_D = \omega_F \cdot r_F$$

and

$$\frac{\omega_F}{\omega_D} = \frac{r_D}{r_F} = \frac{D_D}{D_F}$$

where D_D is the diameter of the driver and D_F is the diameter of the follower.

If now we leave the imaginary case of a belt of no thickness and face the fact that a belt does have finite thickness, we find that at a contact point, and throughout the range where belt and pulley are in contact, the outer surface of the belt has a greater linear velocity than points lying nearer the surface of the pulley. It is customary in practical machine design to neglect this velocity difference, and the radii or diameters of the pulleys are used in establishing the speed ratio. If the thickness of belt is great, compared with the diameter of the pulleys, such an approximation becomes too crude to serve as the design basis.

It will be noted that when a belt is bent around a pulley (Fig. 156), the

outside fibers must stretch and the inside fibers must contract from their original lengths. The fibers at the center of the belt will remain unchanged in length, the exact location of these "neutral-length" fibers varying slightly with the shape of the cross-sectional area of the belt.

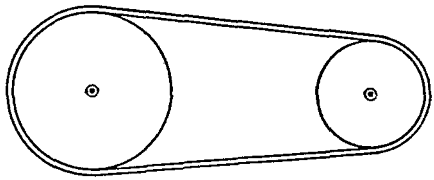


FIG. 156

If it is assumed that the fibers of unchanged length lie at the middle of the belt (a very reasonable assumption for the usual belt), the exact speed ratio for the belt drive becomes

$$\frac{\omega_F}{\omega_D} = \frac{r_D + \frac{t}{2}}{r_F + \frac{t}{2}}$$

where t is the thickness of the belt. It will be noted that only the ideal conditions for velocity transmission have been considered. There is some belt slip in actual practice, and the speed of the follower may be reduced as much as 5 per cent from the ideal speeds presented in the equations.

In establishing the other part of the velocity property—direction—let us note that when the belts are mounted "open" as in Fig. 41 the pulleys will have the same direction of rotation. When "crossed" as in Fig. 42, the pulleys will rotate in opposite directions. When a flat belt is crossed, the belt is twisted "half around" as in Fig. 42 in order that the same face of the belt will always be the contacting surface. This method of mounting also serves to prevent the two parts of the belt which cross from touching since they now oppose parallel flat surfaces as they pass.

While our concern is with the kinematic properties of mechanisms, and the study of pure kinematics precludes the investigation of the forces which cause motion, there will occasionally arise factors of import which must, for their full appreciation, rest upon backgrounds of force system. We shall, therefore, abandon a dogmatic adherence to the story of motion itself whenever we may enhance our understanding of motion by considering forces which, perhaps unexpectedly, affect the motion. Such cases arise for example when secondary incidental or accidental forces are encountered which cause the bodies in motion to deviate from the paths they would take were only the ideal geometrical conditions of intended paths to be followed.

A case in point is that of the belting we are now discussing.

The material—like leather—of which flat belts are composed has physical properties which affect the design of a belt drive. For example, in the

front elevation of the belt shown in Fig. 157-a, we find that the belt is extremely stiff and resists being bent into such a position as that suggested by the dotted lines.

In the side view (Fig. 157-b) we note that the belt is very flexible, and may be bent into any desired shape, as, for instance, in wrapping it around a pulley.

The resistance of the belt to being bent laterally (Fig. 157-a) is called its *lateral stiffness.*

A segment a–d of a belt is illustrated in Fig. 158.

When this segment is bent laterally, as shown, we may assume that the bend is composed of a series of small chordal distances like a–b, b–c, etc.

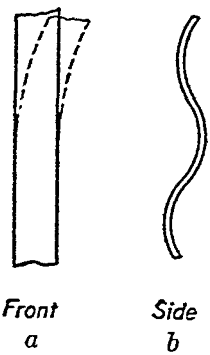


Fig. 157

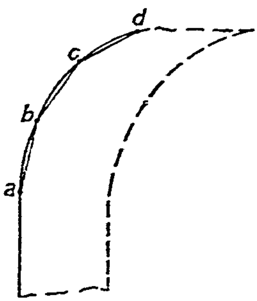


Fig. 158

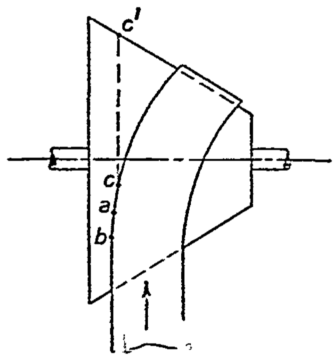


Fig. 159

The assumption is warranted since increasing degrees of lateral stiffness will tend to cause the belt more strongly to resist being bent out of a straight line; in addition, belts while driving are in tension and the effect of tension applied to a bent member is to pull the bent portion into a straight line.

As the belt is led onto a pulley, as in Fig. 159, where the pulley is conical, the belt is bent so that it lies flat on the contacting elements of the cone. When point a arrives at the contact level indicated as point c, it establishes contact with the pulley which is rolling contact within the limits of frictional resistance, and point a will then travel with the mated point c of the pulley with no slip. It will therefore follow the path Cc', and the belt has begun to climb to the greater diameter of the cone. Since the belt remains bent as it climbs, and is laterally stiff, point b must meet the contact level to the left of the position at which point a arrived and will then travel in a path parallel to that of point a, but slightly to the left of it.

Succeeding points as they come into contact will mate with points on the pulley and travel in successive paths which are always to the left of the paths of their predecessors.

Then the belt is constantly climbing toward the greatest diameter of the cone, and will slip off the pulley.

It should be noted that if we call the part of the belt which is drawing near to the pulley the approaching side, and that which is leaving a pulley the receding side, then the only section which plays a role in this "climbing of the pulley" is the approaching side. This is due to the establishment of the rolling contact which carries the moving points, like a, up and over the pulley and results in the climbing effect. When a belt is receding its contact with the pulley is broken, and the motion of receding points is no longer directed by the rolling contact.

If cylindrical pulleys, incidental to their being placed in position, are slightly out of alignment, the belt will be bent laterally as in the preceding case, and may slip off the pulley. Or accidental thrusts upon the belt may cause lateral bending and cause a similar tendency to depart from the pulley.

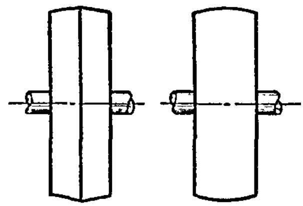


Fig. 160

These tendencies may be corrected by taking advantage of the climbing tendency due to the bending. The pulley, if "crowned" as in Fig. 160, will have its greatest diameter at its central plane, and the belt, if accidentally displaced to either side, will climb back to the center and cling there.

Law of Belting. Another example of the influence of lateral stiffness upon belt drives is encountered in cases like that of Fig. 161, where two cylindrical pulleys are mounted so that their axes are not parallel.

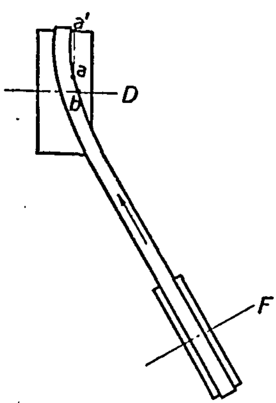


Fig. 161

As point a approaches pulley D and establishes rolling contact with it, point a will then rotate in path aa' around the pulley. The next succeeding point b, due to the lateral stiffness, is trailing behind a and arrives at the contact level to the right of the point at which its predecessor, point a, met it. Then point b will travel, with rolling contact, along a path parallel to a–a' but slightly to the right of it. Succeeding points will each arrive at the contact level to the right of the preceding one and the belt will eventually run off the pulley.

To avoid this mishap, the drive must be designed so that a belt is led squarely onto the pulley, that is, with no bend in the approaching side. This may be accomplished by moving the pulley D along its own axis, as has been done

in Fig. 162, until the central plane of pulley D passes through the point g at which the center of the belt left the lower pulley F.

Now the bend of the belt has been removed from the approaching side of D, and placed on the receding side of F, where there is no harmful effect.

This necessity in pulley location is summarized as the fundamental law of belting: *A belt must be delivered to a pulley so that there is no bend in the*

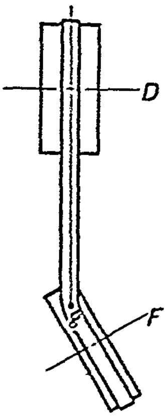


Fig. 162

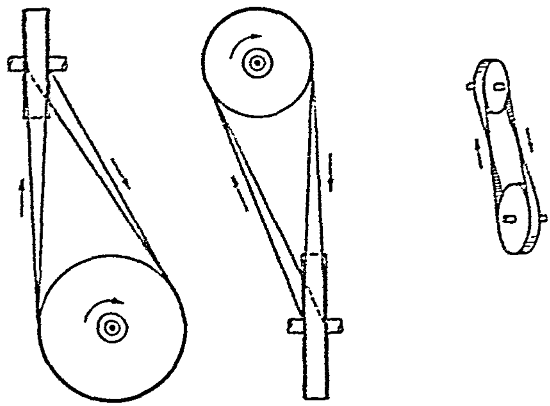


Fig. 163

approaching side. An equivalent statement of this requirement is that *the central plane of the pulley which a belt is approaching must be so located*

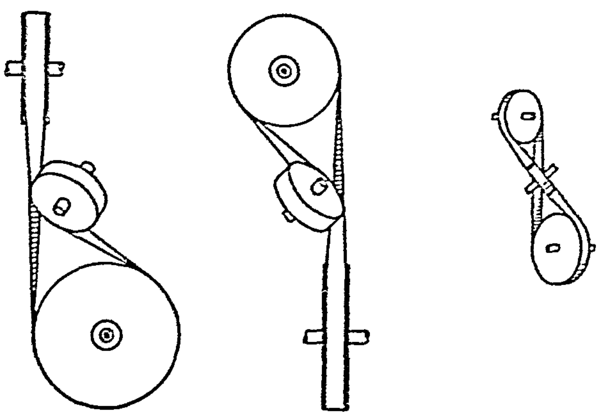


Fig. 164

that it includes the point at which the center line of the belt left the previous pulley.

An illustration of a belt drive is shown in Fig. 163. Let us check this drive to note the application of the fundamental law of belting. Here we

find two shafts which are neither parallel nor intersecting, connected by belting. As long as the drive is operated in the direction shown by the arrows, the pulley arrangement lives up to the fundamental law—the midpoint of the approaching side of the belt does lie in the central plane of the pulley, and the belt will remain in place on the pulleys. If, however, we reverse this drive, we have violated the governing law, and the belt will run off the pulleys. To make such a drive reversible, we may place an intermediate or guide pulley between the driver and follower pulleys, as in Fig. 164.

The guide pulley must be located to direct the belt so that, operating in either direction, it is led onto every pulley so as to conform to the law of belting. The speed ratio remains unaffected by the insertion of the guide pulley in the drive.

PROBLEMS

204. A shaft rotating at 300 r.p.m. is to drive a parallel shaft at 240 r.p.m. Driving pulley = 24 in. diameter. Find the diameter of the follower pulley if (a) the thickness of the belt is neglected, and (b) if a belt thickness of $\frac{3}{8}$ in. is considered.

Ans. (a) 30.0 in.; (*b*) 30.1 in.

205. Two parallel shafts 8 ft. apart are to be connected by a belt so that they run in the same direction. The speed ratio is 5 : 1, and the diameter of the driving pulley is 30 in. Calculate the length of belting required, neglecting the thickness of the belt.

206. Solve Problem 205, if the shafts are to run in opposite directions.

207. A machine is to be driven at a speed of approximately 400 r.p.m. by a motor whose speed is 720 r.p.m. The belt speed must not exceed 3000 f.p.m.

Design the drive if pulleys are to be selected from a manufacturer's list giving even-inch diameters. Neglect belt thickness and slip.

62. Variable Speed Belting Drives. An inspection of the speed ratio

$$\frac{\omega_F}{\omega_D} = \frac{D_D}{D_F}$$

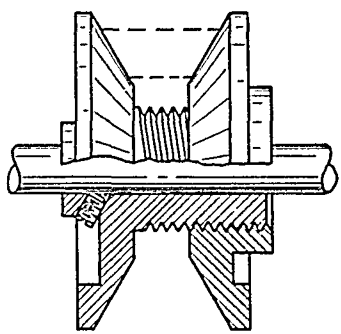
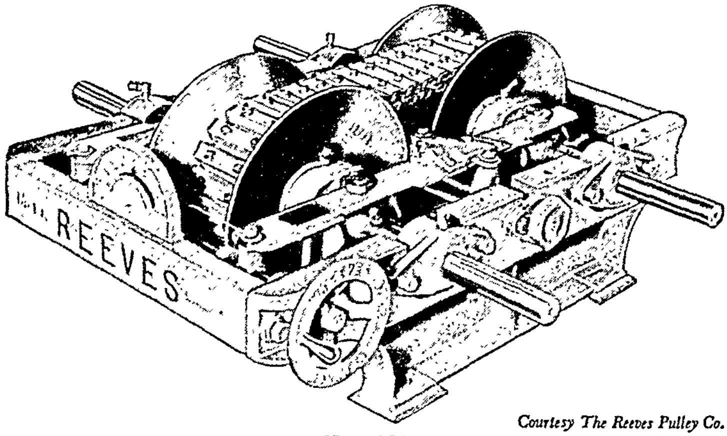


FIG. 165

shows that fixed diameter pulleys on the driver and follower shafts establish a fixed speed ratio. Flexibility in the speed ratio may be obtained by substituting for the fixed-diameter pulleys wheels which are split, like those which were shown in Fig. 44. Such pulleys present an opportunity of changing the ratio of diameters. By moving the two halves of the pulley further apart, the belt will drop lower into the valley formed by the two conical half-pulleys. As it drops to the new position, a new smaller radius is established. If the other pulley of the drive

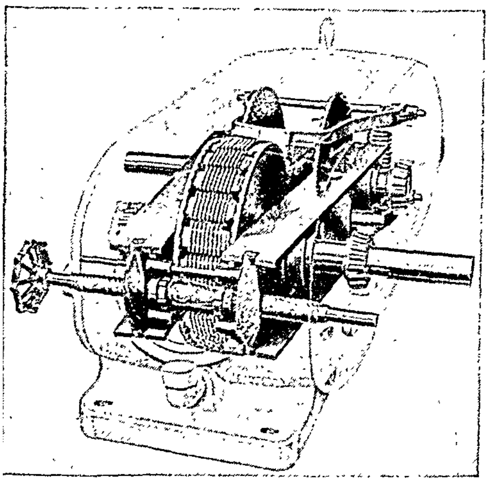
is similarly split and capable of having the distance between its halves changed, its radius may be increased as the radius on the other pulley is



Courtesy The Reeves Pulley Co.

Fig. 166

decreased. The speed ratio $\frac{\omega_F}{\omega_D} = \frac{r_D}{r_F}$ will change and the drive becomes an effective one where an easily available change of speed ratio is wanted. Incidental to the basic drive, auxiliary mechanisms must be supplied to move the half-pulleys apart, and to insure retaining proper tension in the belt.



Courtesy The Link-Belt Co.

Fig. 167

Applications of this basic kinematic principle are found in a number of speed changers. The most simple is the adjustable sheave of Fig. 165. This sheave may be set in a number of positions but is, of course, not flexible when once set.

A flexible type is illustrated in Fig. 166.

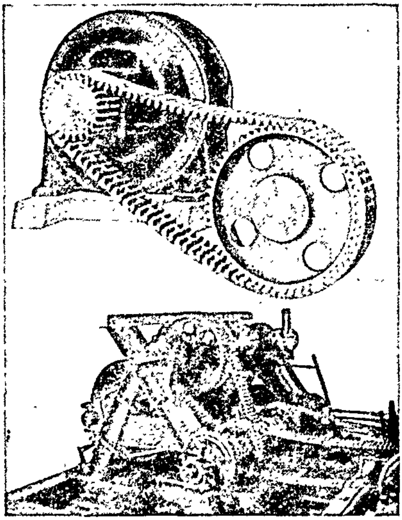
Another type is shown in Fig. 167. The conical pulleys are joined by a chain of unique design. Radial teeth are cut in the faces of the wheels, and adjustable laminated teeth project beyond the side of the chain to engage these teeth positively. This drive, then, is no longer dependent upon friction, but furnishes positive motion.

63. Chain Drives. Chain drives are illustrated in Fig. 168.

Kinematically, the chain drive is equivalent to the bending of a rack around a pinion (see Art. 65), with the attendant advantages of positive drive over other flexible connectors; and, in the case of roller-type chains, the advantage of rolling contact over the sliding contact of rack and pinion.

The speed ratio is fixed by the numbers of teeth on the toothed wheels, or sprockets, and

$$\frac{\omega_F}{\omega_D} = \frac{T_D}{T_F}.$$



(Upper) Courtesy The Link-Belt Co.
(Lower) Courtesy Diamond Chain & Mfg. Co.
FIG. 168

The shape of sprocket teeth has been standardized, and is the result of compromise between the ideal conditions which would prevail if the sprocket and chain remained unchanged in use and actual operating conditions. Chains will stretch and wear, although in the case of the roller chain the advantages of rolling contact minimize these evils.

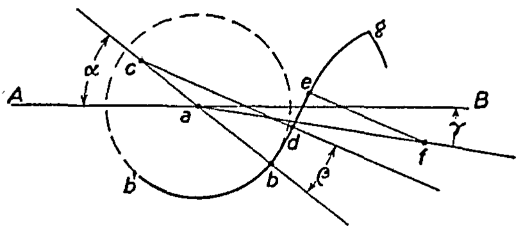


FIG. 169

The standard tooth form for a roller chain sprocket is given in Fig. 169.

$$D = \text{Roller diameter}$$
$$T = \text{Number of teeth}$$
$$D' = 1.005\,D + 0.003''$$
$$\alpha = 35° + \frac{60°}{T}$$
$$\beta = 18° - \frac{56°}{T}$$
$$\gamma = \frac{180°}{T}$$

With a as center draw arc $bb'f$ of radius $ab = \frac{1}{2} D'$.
Draw line bc at angle α with AB.
Make $ac = 0.8\ D$.
Draw line cd at angle β with cb.
Draw line de perpendicular to cd.
Draw line af, making angle γ with AB, and locate point f with $af = 1.24\ D$.
Draw line ef parallel to cd.
With f as center and radius fe, draw arc eg.

64. Flexible Connectors in Hoisting Tackle. Flexible connectors, like ropes, may be used with pulleys to derive mechanical advantage in such applications as hoisting tackle. The purpose of such devices is to overcome a great resistance acting through a comparatively small distance by means of a relatively small force acting through a large distance.

FIG. 170

Mechanical advantage is the ratio of the force overcome to the force that would have to be applied if we disregarded friction. It may be established, from the kinematic viewpoint, by dividing the velocity of the point of application of the lifting force by the velocity of the point of application of the lifted weight. While this method is slightly more cumbersome than the free-body and force relationships generally employed, the velocity method is of advantage in the study of mechanical advantage in many mechanisms.

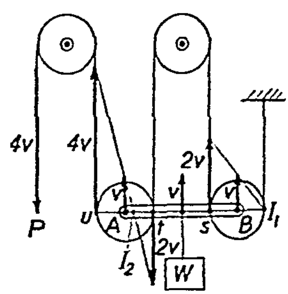


FIG. 171

The block and tackle shown in Fig. 170 is an ordinary form of hoist.

Figure 171 is a diagrammatic sketch of the hoisting tackle of Fig. 170 which is a more convenient representation to show clearly the velocity relationship.

If the bar AB, from which the weight is suspended, is given an upward velocity, v, of translation, all points like A and B will have velocity v.

I_1 is the instantaneous axis of the right-hand lower pulley, and point s of that pulley will have velocity $= 2\,v$. This velocity is transmitted along the rope to point t. Since t has velocity $2\,v$ and A has velocity v, we may locate the instantaneous axis, I_2 of this pulley, and the velocity of point u becomes $4\,v$ by the proportion of distances from the instantaneous axis. This will be the velocity of the point of application of the lifting force P.

Then the mechanical advantage is

$$\frac{4\,v}{v} = 4.$$

Differential pulley-blocks, like that of Fig. 172, are a simple form of hoist using chain as an endless flexible connector. The representation for kinematic analysis is shown in Fig. 173.

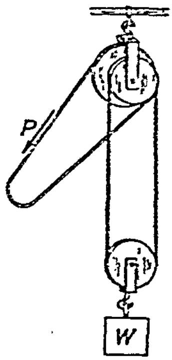


FIG. 172

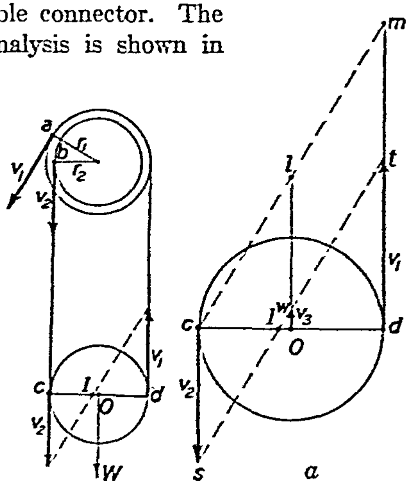


FIG. 173

The upper pulley is a two-step pulley or sheave with diameters of the two steps slightly different.

If point a is given a velocity v_1, the upper pulley has angular velocity

$$\omega_1 = \frac{v_1}{r_1}\cdot$$

Then point b will have linear velocity v_2 and

$$v_2 = \omega_1 r_2.$$

These velocities are transmitted by the flexible connector to the lower pulley.

There we find point c with velocity v_2 and point d with velocity v_1 and the instantaneous axis of the lower pulley is at point I. (See Fig. 173-a.) Then v_3 will be the velocity of point O, and of the weight.

If we draw a line cm from c parallel to st, then cs, lw, and mt are all equal to v_2.

Then, from similar triangles cdm and cOl,

$$\frac{v_1 + v_2}{v_3 + v_2} = \frac{cd}{cO} = \frac{2}{1}$$

and

$$v_1 - v_2 = 2\,v_3.$$

But

$$\frac{v_1}{v_2} = \frac{r_1}{r_2}$$

and

$$v_2 = v_1 \cdot \frac{r_2}{r_1}$$

$$v_1 - v_1 \frac{r_2}{r_1} = 2\,v_3$$

$$\frac{r_1 - r_2}{r_1} = \frac{2\,v_3}{v_1}$$

and

$$\frac{v_3}{v_1} = \frac{r_1 - r_2}{2\,r_1}$$

which as velocity ratio, is the reciprocal of the mechanical advantage. It is usually expressed in terms of diameters D_1 and D_2 of the upper pulley, as

$$\text{Mech. adv.} = \frac{2\,D_1}{D_1 - D_2}.$$

65. Gearing. Definitions and Terminology. The distinction between the friction drive and drives producing positive motion was pointed out in Art. 19. We noted then, in the illustration of the block used as driver transmitting motion to a follower block, the advantage of departing from dependence upon frictional resistance (limited by the impending slip) and progressing to the positive-drive action of projecting tooth and recessed slot.

A pair of rolling cylinders furnishes another basis of opportunity to secure positive drive by substitution of toothed surfaces for the plain cylindrical surfaces which must depend upon friction for successful transmission of power.

We must, however, be cautious in effecting this substitution. In the case of the rolling cylinders, the speed ratio $\frac{\omega_F}{\omega_D}$ is a constant quantity. This fixity becomes disturbed only when the radial distances to the contacting surfaces are changed, or when slip occurs.

The constant speed ratio must be preserved when we change from the use of smooth cylindrical surfaces to notched or toothed surfaces. We must

not find upon substituting the toothed surface that we have done violence to the fixity of the speed ratio, and that our toothed driver operating at constant speed causes the follower to have a variable speed. The design of the profile of the teeth—the geometry of gear teeth—rests upon this fundamental principle; *constant speed of the driver must produce constant speed of the follower*, and the *speed ratio must be the same at every instantaneous position of contact.*

As we enter the discussion of design which will fulfill these conditions, the terminology of gearing must be familiar to us.

The nomenclature and definitions may be most solidly established by keeping as a definition background the pair of rolling cylinders, since the entire procedure of establishing the proper shape of the teeth is a search to retain the basic speed ratio of these rolling cylinders.

We need not operate in three dimensions since conditions of shape and motion are identical in all the planes parallel to each other and perpendicular to the axis of rotation in the case of rolling cylinders or their refinement into gears. We shall therefore confine our studies to a single plane of motion.

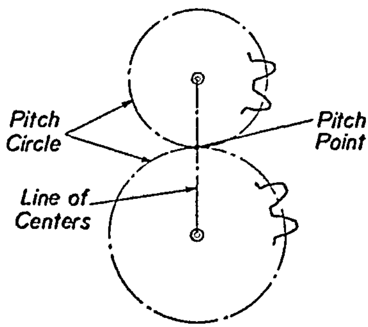


FIG. 174

The rolling cylinders then become rolling circles, and are called the *pitch circles.* The line joining their axes is the *line of centers,* and the point of contact of the pitch circles is the *pitch point.* These are illustrated in Fig. 174. The diameters of pitch circles are called *pitch diameters.*

The pitch circles fix the speed ratio

$$\frac{\omega_F}{\omega_D} = \frac{D_D}{D_F}$$

where D_D is the pitch diameter of the driver

and D_F is the pitch diameter of the follower.

Using the pitch circle as a basis we form the teeth, and the terms associated with the tooth form are illustrated in Fig. 175.

The outermost circle of the gear is called the *addendum circle,* and the radial distance from the pitch circle to the addendum circle is the *addendum.*

The circle which is found at the bottom of the space between teeth is the *dedendum circle,* and the radial distance from the pitch circle to the dedendum circle is the *dedendum.*

The dedendum is the sum of two distances. When another gear is in mesh with that shown in Fig. 175 its addendum will project into the space between teeth of that gear to a depth bounded by a circle shown in the figure as the working-depth circle. The *working depth* is the radial distance from the addendum of a gear to the working-depth circle. Proceeding

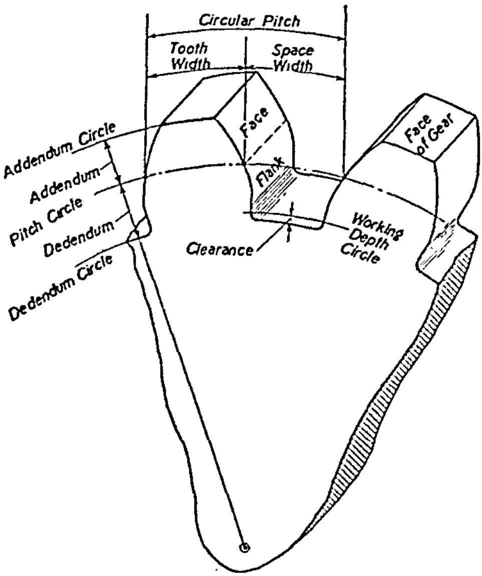


FIG. 175

toward the center of the gear a space is left for the *clearance* between the dedendum circle of the gear and the addendum circle of the mating gear.

The *circular pitch* is the distance, measured along the pitch circle from a point on one tooth to the corresponding point on the next tooth. It thus contains both *tooth width,* and the *tooth space,* or width left between teeth, both of which are measured along the pitch circle. The circular pitch is equal to the circumference of the pitch circle divided by the number of teeth on the gear, or

$$\text{C.P.} = \frac{\pi D}{T}$$

where D is the pitch diameter, and T the number of teeth.

The *face* of the tooth is the surface lying, as shown in the figure, between the pitch and the addendum circles.

The *flank* of the tooth is the surface which lies between the pitch and dedendum circles.

The *diametral pitch*, abbreviated D.P., is the ratio of the number of teeth, T, to the pitch diameter, D, or

$$\text{D.P.} = \frac{T}{D}.$$

When gears are in mesh, as illustrated in Fig. 176, the tooth width cd of one gear may be less than the space between teeth ab of the mating gear. There is then some clearance between the teeth. Such clearance, always measured along the pitch circle, is called *backlash*. While in general the provision of backlash has undesirable by-products like the creation of noise or hum, some must be provided to leave room between teeth for lubricant and to allow room for expansion of the teeth with rise in temperature.

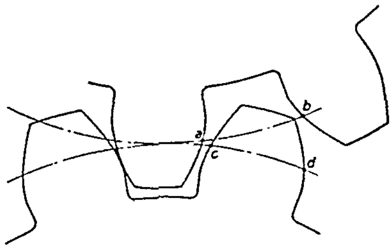


FIG. 176

The relationship between the two types of pitch, circular and diametral, is a convenient one.

Since

$$\text{C.P.} = \frac{\pi D}{T}$$

and

$$\text{D.P.} = \frac{T}{D}$$

then

$$\text{D.P.} \times \text{C.P.} = \frac{T}{D} \times \frac{\pi D}{T} = \pi$$

or

$$\text{D.P.} = \frac{\pi}{\text{C.P.}}.$$

We require one other basis of definition.

When two teeth of mating gears are in contact, as at a in Fig. 177, a line may be drawn from the point of contact to the pitch point. This line is called the *pressure line* or line of obliquity.

The angle θ formed by the pressure line and a normal to the line of centers of the gears at the pitch point P (see Fig. 177) is called the *pressure angle*, or, sometimes, the angle of obliquity.

The locus of all the points of contact is the *path of contact*. This locus may be straight or curved, but we shall find that it must always pass through the pitch point.

both gears, the amount of pitch-circle arc on each gear passing by the pitch point is the same. Since this arc has already been called the arc of action we conclude that *arcs of action on two mating gears are equal.*

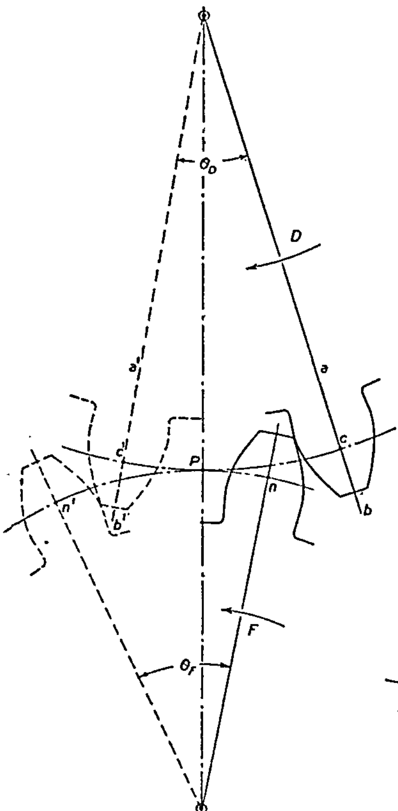


FIG. 178

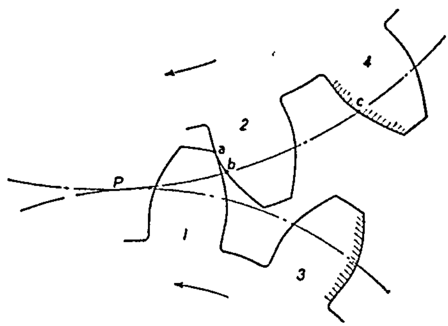


FIG. 179

When two gears are in mesh there must be no interruption of the contact, for there would then be an interruption in the fixity of the speed ratio. We cannot, therefore, permit contact between one pair of teeth to cease before the succeeding pair have come into contact.

In Fig. 179 teeth *1* and *2* are just starting the contact at point *a*. During the time that they remain in contact, the amount of pitch circle arc which will pass the pitch point has now been defined as the arc of contact. If, at the instant teeth *1* and *2* are just ceasing their contact, the shaded surfaces of teeth *3* and *4* have not yet reached point *a* and made contact with each other, there will be an interval during which there is no contact between the gears. But point *c* of tooth *4* is at a distance of the circular pitch from point *b*. It is therefore absolutely essential that an amount of pitch circle arc = *cb* = C.P. must pass by the pitch point for the tooth surface containing *c* to arrive at point *a* in time to maintain continuity of contact between the gears. This fact should, therefore, become a rule or governing condition of the gear design; that is, *the minimum permissible arc of action is the circular pitch.*

We should note that when the arc of action is greater than the circular pitch, point c will arrive sooner than necessary to preserve the continuity, and more than one pair of teeth will be in contact at the same time. Such an overlap of action is desirable in that there is a direct influence upon the load to which each tooth is subjected, and more favorable conditions will arise when the total load is distributed over more than one pair of teeth simultaneously.

It is convenient to differentiate between two stages of the contacting action. The contact action from the time that contact starts until contact is at the pitch point is called *approach*. The contact action from the time contact is at the pitch point until contact ceases is called *recess*.

Definitions follow from this differentiation in this fashion:

The locus of all points of contact during approach is called the *path of approach*, and the similar locus during recess, the *path of recess*. The *path of contact*, which is the total locus of contacting points, is then the sum of the paths of approach and recess.

The angles and arcs of action are similarly divided.

The *angle of approach* is the angular displacement of a gear during approach, and the *angle of recess* is the angular displacement of a gear during recess; the *arcs of approach and recess* are the subtending arcs of the angles of the same name.

From the previous discussion of the relationship of arcs of action it follows that when two gears are mating the arcs of approach on both gears are equal, as are the arcs of recess. On one gear, the arc of approach will equal the arc of recess only when all conditions of approach and recess have been made symmetrical with respect to the pitch point.

When gears are used to connect parallel shafts, they are called *spur gears*.

When two gears are in external contact, the smaller gear is called the *pinion* and the larger the *gear*.

When two gears are in internal contact, the smaller gear is called the *pinion*, and the larger the *annular*.

A *rack* is a gear with pitch diameter equal to infinity; that is, the pitch circle is a straight line. The size of pitch diameter is the only distinguishing feature of a rack—it is in all respects a gear conforming to the definitions just outlined for all gears.

It should be noted that throughout the development of our vocabulary of gearing we have been centering our interest in the rolling cylinder background, which has evolved through modification into the toothed cylinder, and the definitions which have been given originate in the pitch circles. This recognition of the role of the rolling cylinder as a background or basis will continuously serve as a solid and comforting fundamental from which to expand our appreciation of the action of the gears.

PROBLEMS

208. Two shafts, D and F, 15 in. apart, are connected by a pair of gears so that they rotate in opposite directions. The speed ratio is 1.5 : 1. The diametral pitch = 5. Find the number of teeth on each gear. *Ans.* $T_D = 90$; $T_F = 60$.

209. *What is the distance between the centers of two gears, D and F, which are in external contact with a speed ratio of 1 : 3; diametral pitch = 2; number of teeth on F = 24?*

210. Two gears are mounted so that they turn in the same direction. The diametral pitch = $3\frac{1}{2}$; the pinion has 38 teeth and the annular has 152 teeth. How far apart are the centers of the gears?

211. Gears D and F are in internal contact. If the pinion, D, has 48 teeth; the speed ratio is 2 : 5; and the distance between centers of gears is 14.4 in.; determine the diametral pitch and the circular pitch of the gears. *Ans.* 2.5; 1.26 in.

212. Two gears in contact, D and F, are turning in opposite directions. The driver has 122 teeth, the follower has 86 teeth. If the circular pitch = 0.628 in., determine the distance between the axes of D and F.

213. Two gears in contact, D and F, are mounted so that they turn in the same direction. Diametral pitch = 2. The driver has 64 teeth, and the follower has 32 teeth. The addendum = 1/D.P. If the path of contact is a straight line and the pressure angle = $14\frac{1}{2}°$, determine the length of the paths of approach and recess.

214. A gear is cut with diametral pitch = 2. Determine the minimum arc of action which this gear may be permitted to have. *Ans.* 1.57 in.

215. Using the data of Problem 214, determine the smallest angle of action with which the gear may be satisfactorily operated if the gear has 16 teeth.

216. A gear of 24 teeth drives a follower of 16 teeth. If the diametral pitch = 1, determine the minimum permissible arc and angle of action for each gear.

217. The speed ratio of a pair of gears in contact is 2 to 4.2, and they rotate in the same direction. If the pinion driver has 20 teeth, and the diametral pitch = $1\frac{1}{2}$, determine the distance between the centers of the gears.

66. Fundamental Law of Tooth Profile. In establishing the vocabulary of gearing, we have spoken of the surface of the tooth without describing the exact nature of the profile, or geometric shape.

If we restate the limitation under which toothed wheels may be permitted to replace the rolling cylinders, we shall have announced the condition governing the shape of the teeth whose contact is replacing the frictional contact of the rolling cylinders. This limitation demands that the teeth must be of such shape that the speed ratio must be at all times equal to the speed ratio of the replaced rolling cylinders. A pair of rolling cylinders, D and F, which in the gears will become the pitch circles, are shown in Fig. 180 where they are in contact at point P, which in the gears will be the pitch point.

The speed ratio is

$$\frac{\omega_F}{\omega_D} = \frac{dP}{fP}.$$

It is this ratio which is to be held constant.

The triangles dPm and fPn are similar.

Then

$$\frac{dm}{fn} = \frac{dP}{fP}.$$

Since we wish to preserve the speed ratio a constant ratio, we must maintain the fixity of $\frac{dP}{fP}$.

Let us now see how this basis enters into the geometry of the tooth shape, by consulting Fig. 181.

Here we find two gear teeth in contact at point c, a point on AB, which therefore becomes the pressure line, conforming to definition.

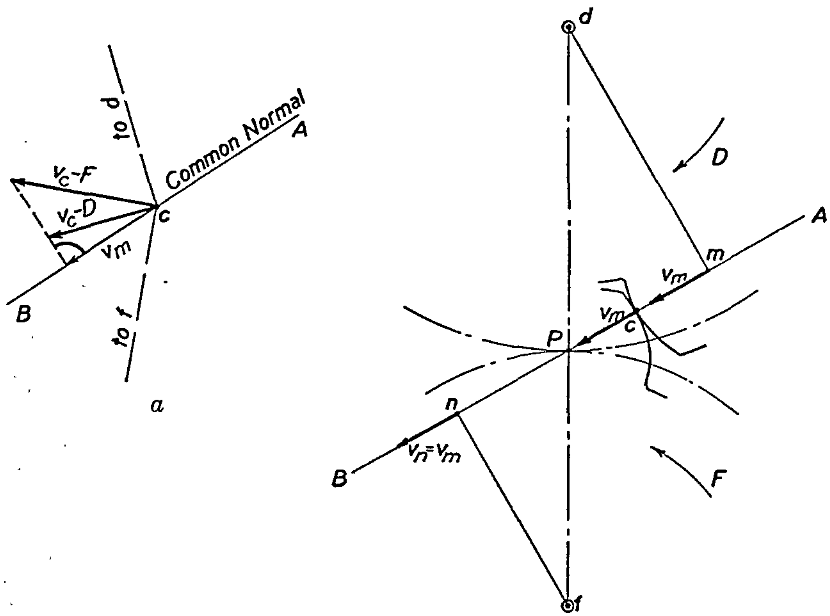


FIG. 181

We shall again trace through the various stages in the transmission of velocity, remembering that the contact is no longer between pitch circles as rolling cylinders but between the gear teeth.

If again lines dm and fn, both perpendicular to AB, are established,

$$v_m = \omega_D \cdot dm.$$

v_m is the orthogonal component of c's velocity in direction AB.

Now let us note, before going through the contact and on to the other body, that when any two curves are tangent to each other they have a com-

mon tangent at the point of contact, and a line normal to this common tangent is a common normal to the curves.

Assuming that we have so designed the tooth surfaces that when they are in contact at c, AB is the common normal, we may proceed to observe the velocity relationships.

Point c, on D, has an orthogonal component of velocity along AB which is equal to v_m. Then point c, on F, has the same orthogonal component of velocity in direction AB. This statement rests, for authority, upon the discussion of Sliding Contact (Art. 50) where we found that when two bodies are in sliding contact the orthogonal component of velocity which is normal to the sliding surface is the same for the contact point on each body. Fig. 181-a shows the complete velocity resolution at point c, at enlarged scale.

Since c on F has orthogonal component equal to v_m, then n on the same rigid body must also have orthogonal component equal to v_m. But, since fn is perpendicular to AB, this orthogonal component is also v_n, the *resultant* velocity of n.

Once again, although we have traced the velocity transmission through a different type of contact between bodies D and F, we have arrived at the same conclusion.

$$\frac{\omega_F}{\omega_D} = \frac{dm}{fn} = \frac{dP}{fP}.$$

Again, if $\frac{\omega_F}{\omega_D}$ is to remain constant, the ratio $\frac{dP}{fP}$ must remain fixed. Now, with gear teeth rather than rolling cylinders forming the contacting surfaces, the ratio $\frac{dP}{fP}$ has only one possible condition under which it can remain fixed; the common normal, at every point of contact, must pass through point P, thus dividing the line of centers into the same portions continually.

This conclusion may be stated as follows: if tooth surfaces are shaped so that at every point of contact the common normal passes through the pitch point, the gears will maintain the desired constant speed ratio. We have observed the underlying velocity conditions which lead to this conclusion for one point, c. This point had no special properties, and statements made concerning it become general statements which hold for any other point.

It is worth while to strengthen this conclusion by some form of check, which has so frequently been called to attention as sound engineering philosophy.

Let us then examine, for comparison, the velocity transmission which would develop if the common normal to the tooth surfaces at some points of contact did *not* pass through the pitch point.

In Fig. 182 we find two tooth surfaces in contact at a. These are so shaped that the common normal at the point of contact does not pass through the pitch point P, of the rolling cylinders, but cuts the line of centers at point S.

Now

$$\frac{\omega_F}{\omega_D} = \frac{dm}{fn} = \frac{dS}{fS},$$

which is not the intended speed ratio. In addition, if such teeth were employed, the common normal to their surfaces when they were in contact later (for example, at point b) would cut the line of centers at such a point as T. Now we have a new and different speed ratio; we are not preserving a fixed speed ratio, and are therefore violating the limitation which was originally imposed; namely, that replacing rolling cylinders with toothed wheels must not do violence to the constant speed ratio.

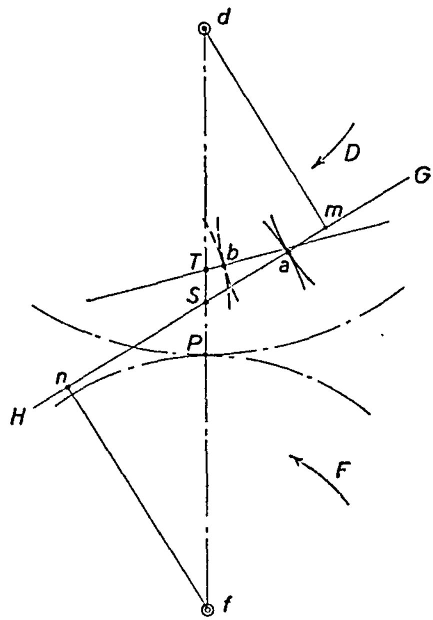


Fig. 182

Let us summarize these observations in a statement which is the fundamental law of tooth profiles: *The common normal to the tooth surfaces at every point of contact must pass through the pitch point.*

67. Sliding Contact in Gear Teeth. The replacement of rolling cylinders by toothed wheels has produced the equivalent velocity properties of the rolling bodies; but the actual contact which has been substituted is a sliding one, except for the one instance when contact is at the pitch point. We may check this observation by a study of the velocity vector resolution at any point of contact other than that of the pitch point.

In Fig. 183 gears D and F are in contact at point a. The velocity of point a on body D is $v_1 = \omega_D \cdot da$ and perpendicular to da. The velocity of the contacting point a on body F is $v_2 = \omega_F \cdot fa$, which is perpendicular to fa.

Since v_1 and v_2 are not identical, there is relative velocity, which is their difference, bc, of Fig. 184. The sense of vector bc is not shown, for we must, to establish the sense of the relative velocity, announce which of the two possible senses we are interested in—the velocity of v_1 relative to v_2, or v_2 relative to v_1. In either event, however, we observe that bc, the difference

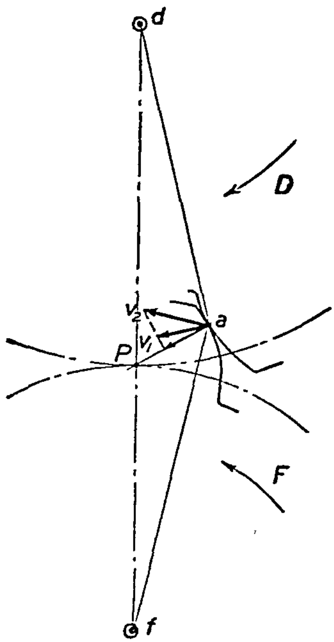


Fig. 183

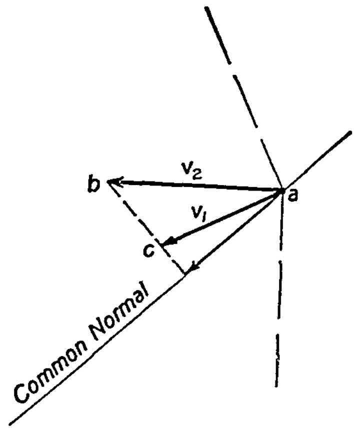


Fig. 184

between v_1 and v_2, is a component which is the rate of sliding. When contact is at the pitch point, there will be identical values of v_1 and v_2 and therefore no relative velocity; at the pitch point, and only there, the gears will be in such contact that the conditions of pure rolling are fulfilled.

68. Conjugate Curves are those which conform to the fundamental law of gearing.

It is quite possible to start with any given shape of tooth and obtain a conjugate curve which will mesh properly with the given one.

For example, in Fig. 185 two pitch circles D and F are shown and on D is found a plate, which may be of any shape. It is desired to establish the conjugate curve on F which will mesh with the given one on D so that the fundamental law of gear tooth profile will be fulfilled; in other words, the plate on D is to act as a gear tooth properly meshing with a plate or tooth on F so that the pitch circles will behave like cylinders in pure rolling contact at P, the pitch point.

We start by considering the definition of conjugate curves—curves which are paired so that whenever they are in contact a common normal must pass through the pitch point.

Then we erect at any point like point *1* (on the surface of the given plate

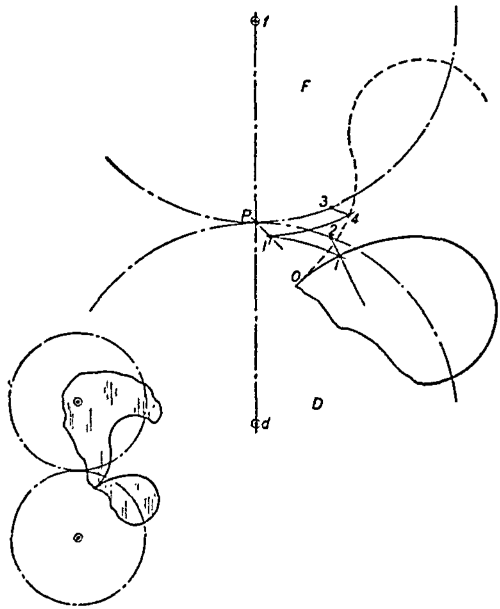


FIG. 185

on D) a line 1–2 which is normal to the plate surface, with point 2 at the intersection of the normal and the pitch circle. If rolling cylinder D is rotated counter-clockwise until point 2 arrives at the pitch point, P, then point 1, moving in a circular path about axis d, will arrive at position $1'$, with distance $1\text{–}2 = 1'\text{–}P$.

Now the normal to one curve passes through the pitch point. Referring to the definition, at this instant the two curves should be in contact, at point $1'$.

Now let us consider rolling cylinder F. During the time that point 2 moved to the pitch point, a point on the surface of rolling cylinder F, which is turning clockwise, will also have moved to the pitch point.

This point will be at a distance away from P, measured on the circumference of pitch circle F, equal to arc P–2, for we are to maintain pure rolling contact.

Such a point is 3, on the surface of F, with arc $P\text{–}3 =$ arc $P\text{–}2$.

When both points 2 and 3 are at the pitch point, line P–$1'$ becomes the common normal, and point $1'$ the point of contact. To obtain the curve on F we return point 3 from P to its starting position (the one shown as 3 in the figure). At the same time the end-point $1'$ of the common normal, which is now a line of body F, rotates in a circular path about axis f. When

3 arrives at its starting position, point *1′* of the common normal is at position *4* with distance *3–4* equal to *P–1′*, or, in turn *1–2*.

Then point *4* is a point on the surface of a curved plate on rolling cylinder *F* which will be conjugate to the given plate on *D*.

If we repeat the process used in obtaining point *4* for a sufficient number of points and draw their locus, we shall have completed the design of the conjugate plate. In Fig. 185 the conjugate curve on *F* is shown in dotted outline.

In common with all plotted curves the conjugate plate will be an approximation. Refinement of its shape may be made by taking a greater number of points on the surface of *D* and obtaining a closer approximation of *F*'s shape, but the exact solution is impossible graphically, since only an infinite number of plotted points would yield the precise shape.

To insure clear understanding of the action, let us check the derived shape against the fundamental law of gearing which governs the shape of conjugate curves.

We return, therefore, to Fig. 185. In the position shown in the diagram we find contact taking place at *O*.

We may imagine the driver, *D*, to be rotated counter-clockwise until line *1–2*, remaining normal to the surface of the given plate at *1*, arrives in the contact position *P–1′*.

During this time line *3–4* has been rotating and, since the contact between pitch circles is intended to be pure rolling contact, *3* will arrive at *P* at the same instant that *2* does. Normal *3–4* will now coincide with normal *1–2*, both merging as line *P–1′*.

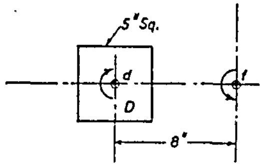
The curves of both plates are in contact at point *1′*. When two curves are in contact they have but one common normal—line *1–2* was originally made normal to one curve—then line *3–4* which at contact coincides with it must be normal to the conjugate curve. Since all points of the curve on *F* are established in identical manner as point *4*, the entire curve is a conjugate curve to its mate on *D*.

We have therefore satisfied the demands of the fundamental law: when the two curves are in contact, their common normal does pass through the pitch point, and these curves are conjugate.

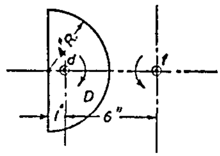
PROBLEMS

Problems 218-220. Two parallel shafts, *d* and *f*, have equal angular velocities of opposite sense. A plate, *D*, shown in the figure is mounted upon shaft *d*, and maintains contact with a conjugate plate, *F*, mounted upon shaft *f*. Design the conjugate plate on *f*, for a 180° rotation of *D*.

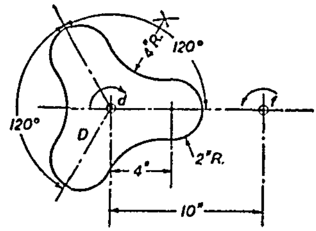
Determine the rate of sliding when plate *D* has turned through an angle of (a) 30°, and (b) 48° from the starting position shown in the figure, if the angular velocity of *D* is 1 radian per sec.



PROB. 218



PROB. 219



PROB. 220

69. The Involute Tooth Profile. The previous article established the possibility of developing a conjugate tooth action when we start with any given tooth profile.

Since the form of the first tooth is arbitrary and its mate is designed to properly mesh, there is theoretically an infinite choice of curves available. In practice, however, while the geometrical properties may be satisfactory, the problems of production, of satisfactory distribution of the frictional effects arising from the force system, and of supplying a strong tooth preclude the use of many forms.

As the design, production, and use of gears have evolved, certain families of geometrical curves have come into prominence.

The most extensively employed form is the *involute* tooth, which possesses characteristics that have established its preeminence to such an extent that it is the only form which may now be called a universal type. As in all other fields, no factor arises to a position of supremacy in mechanisms unless it has marked advantages which permit it to replace and succeed its rivals. Before entering a detailed discussion of the geometry and kinematics of involute gearing let us enumerate the advantages which this system possesses.

First, the path of contact in involute gears is a straight line, and the pressure angle constant. This is not an advantage, of course, if it serves only to simplify design calculations, as it does, for that would become an advantage only to the lazy designer, rather than to the action of the gears themselves. A constant pressure angle, however, is advantageous in that the components of force which have an influence upon the gears and upon the bearings in which they are mounted are constant in nature rather than vibratory, which would be undesirable. In addition to this property, the magnitude of the pressure angle may be held low, in itself advantageous in preventing unduly large thrust in the direction of the line of centers, and consequent excessive load on the bearings.

Second, the distance between centers of a pair of involute gears can be changed without disturbing the constant nature of the speed ratio. This is an extremely important feature, for an alignment given before the loading

is applied is always changed by the deflection resulting from the load, and later by wear of the bearings. In the involute system an adaptation of the geometry underlying this feature makes it possible for two or more gears of different numbers of teeth mounted on the same axis to mesh with the same driving gear.

Third, the geometrical shape is an aid in production, for the tools necessary to form this profile are more easily developed than those for other forms. For example, the rack of the involute system has plane surfaces for its tooth profiles. These are manufactured readily and accurately, and may be converted into tools for cutting teeth in pinions.

The involute of a circle is a definite property of that circle. It may be best visualized by considering the illustration of Fig. 186. If the cord, fastened to the cylinder at A and kept constantly taut, is unwound from the circumference, any point, like B, describes a curve called the *involute of the circle.* If a larger circle is used as a base for the wrapped cord, the involute will be increasingly flatter and we can note that as we increase the radius of this base circle the involute, ever growing flatter, approaches

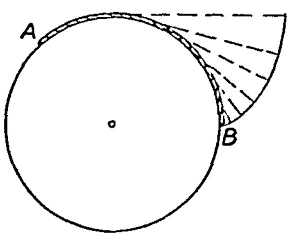


Fig. 186

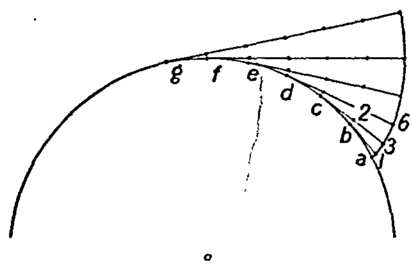


Fig. 187

a straight line. In the limit, with a base circle of infinite radius, the involute *is* a straight line. This case will occur in the rack.

The graphical method of drawing an involute to a given base circle is illustrated in Fig. 187.

Small equal distances, like ab, bc, etc., are first stepped off with the dividers on the circumference. At each point a tangent is drawn to the circle. Now we lay off distances, like b–1, c–2, 2–3, equal to the original divider setting ab, on each of the tangents, proceeding as shown in the figure. The locus of the end points, like a, 1, 3, 6 is the involute of the circle.

70. Involutes as Gear Tooth Profiles. The involute is used as a tooth profile. We must survey its properties to assure ourselves that such a curve conforms to the fundamental law of tooth profile. A pair of pitch

circles and an intended pressure angle θ are shown in Fig. 188. The diameters of the pitch circles are fixed by the desired speed ratio, and the pressure angle, which may theoretically be of any value, is fixed by present standards of good practice at either 14.5° or 20°. These values have a long history of development, and the experience of the past has crystallized into the choice of these two. It is not pertinent to our kinematical discussion to pursue an explanation of the relative merits of the different systems,

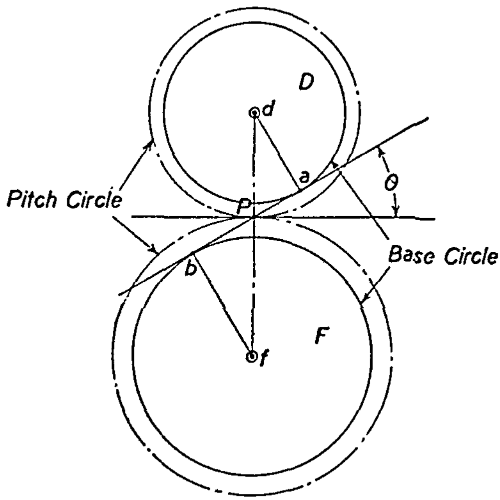


FIG. 188

and we shall accept these two values as being justified by their adoption as standards.

With pressure angle fixed we proceed to establish two circles, concentric with the pitch circles and tangent to the pressure line at a and b. It is these circles whose involutes will become the tooth profiles and they are called the *base circles* of the gears.

The function of the base circles may be clarified by pointing out the analogy between base circles and pressure line, and pulleys and crossed belt. The transmission of motion through the crossed belt has already been discussed.

In the case of the involute gears, the teeth are so designed that contact is continuously along the same pressure line. This contact transmits the drive to the follower in directly analogous fashion to that of a belt mounted upon pulleys. We may note that while the kinematic analogy is direct, there is distinction as far as force system is concerned, for the belt is a tension member and cannot be used to transmit pressure, whereas the direct

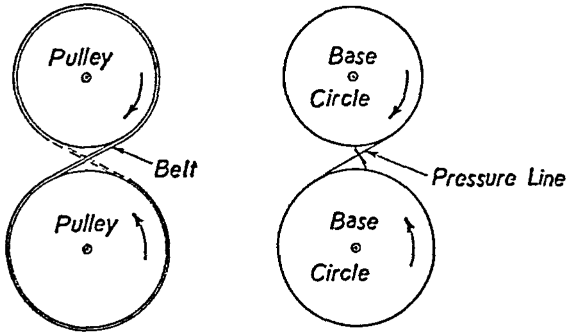


FIG. 189

contact of the tooth surfaces is a push or pressure contact. Figure 189 makes the comparison graphically.

The speed ratio of the base circles, it will be noted from the analogy, is obtained in the same fashion as that of belt and crossed pulleys.

Then

$$\frac{\omega_F}{\omega_D} = \frac{D_{B.D.}}{D_{B.F.}}$$

where $D_{B.D.}$ is the diameter of the base circle of the driver and $D_{B.F.}$ is the diameter of the base circle of the follower.

This speed ratio of the two gears must be identical with that found in Art. 66 where the pitch diameters were employed, for the pitch and base circles are concentric circles of the same rigid body.

If we examine Fig. 188 we note that triangles daP and fbP are similar. Therefore

$$\frac{\omega_F}{\omega_D} = \frac{dP}{fP} = \frac{da}{fb}.$$

These conclusions have rested upon an assumption that when involute teeth are in contact, all points of contact lie in the same pressure line. In addition, we have assumed in drawing them that the fundamental law of tooth profile would be fulfilled by the involute teeth.

If we return to the concept of generating the involute by unwrapping a taut cord from the base circle we shall find our assumptions to be valid. Fig. 190-a will assist us.

First, we note that the straight part of the cord itself is always normal to the involute and ever tangent to the base circle. This is true of both the upper and lower gears of Fig. 190-a.

If now we bring the two gears together as in Fig. 190-b until the tooth profiles touch, they must have, like all contacting curves, *one* common tangent. The line *bc* is normal to the common tangent. Line *st* is also normal to the common tangent, and points *c* and *s* have become one point in space. Then *tscb* is a continuous line and is the common normal to the tooth surfaces. This line is tangent to both base circles. Then it passes through the pitch point *P*, for the base circles whose involutes are making contact were originally fixed as circles tangent to one line (the pressure

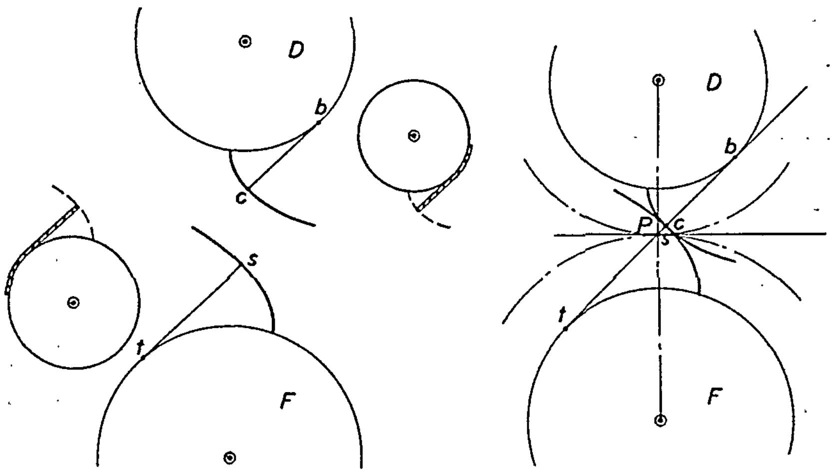


FIG. 190

line) passing through the pitch point. As the gears revolve, different points of the respective involutes will become the contacting points. They will continue to meet, however, on the same pressure line, for whenever these involutes are in contact, they have a common normal which is a continuous line, and this common normal, the straight part of the unwrapped generating cords, is always tangent to the two base circles. Only one line, and that one the straight pressure line, can continuously fulfill this function. Then we have found that involute curves, when used as gear-tooth surfaces, obey the fundamental law of tooth profile. We have also found that the path of contact of such teeth is a straight line, and the pressure angle is constant.

71. Involute Gear Teeth. Now that the geometric nature of the contacting surface of the teeth has been established, we may proceed to complete the entire tooth outline. This may be done most effectively by using

specific data, and developing from them the geometric design of the desired gear as an illustrative example.

It is, then, desired to draw an involute gear with:

Diametral pitch = 1
No. of teeth = 12
Pressure angle = $14\frac{1}{2}°$

The tooth dimensions will be fixed by the American Standard table which follows.

Pressure Angle	$14\frac{1}{2}°$ (*Full Depth*)	20° (*Stub Tooth*)
Addendum	$\dfrac{1}{\text{Diametral pitch}}$	$\dfrac{0.8}{\text{Diametral pitch}}$
Dedendum	$\dfrac{1.157}{\text{Diametral pitch}}$	$\dfrac{1}{\text{Diametral pitch}}$

Then the pitch diameter = 12 in.

$$\text{Addendum} = \frac{1}{\text{D.P.}} = 1 \text{ in.}$$

$$\text{Dedendum} = \frac{1.157}{\text{D.P.}} = 1.157 \text{ in.}$$

Figure 191 is the drawing of the teeth of the gear. Starting with center O

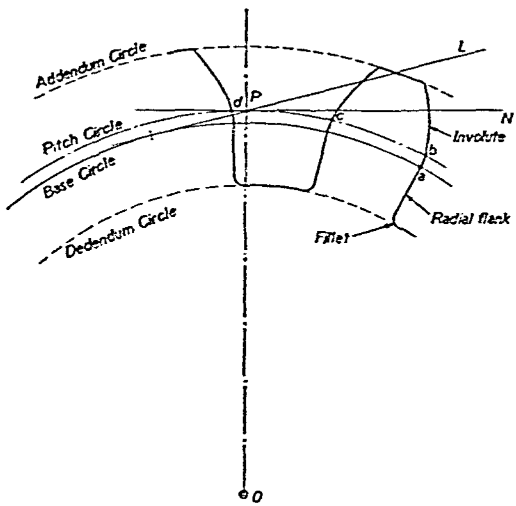


FIG. 191

we draw a pitch circle of radius = 6 in., and line OP to establish a pitch point P.

At P we erect a line PN, normal to OP, and draw the pressure line PL through P making an angle of $14\frac{1}{2}°$ with normal PN.

With O as center, a circle is drawn tangent to the pressure line PL. This is the base circle. At any convenient point, a, on the base circle an involute of the base circle is next constructed. (The graphical method of drawing the involute is given in Art. 69.)

Now we lay off along the pitch circle, starting at point b, an arc distance bd equal to the circular pitch, which in this case is equal to $\frac{\pi}{\text{D.P.}} = \frac{\pi}{1}$ $= 3.14$ in. For greatest accuracy the circumference should be split with the dividers into the same number of arc segments as the required number of teeth.

The arc bd is now divided into two equal parts $bc = cd$. Through c we draw an involute of the base circle, reversed in direction from the involute at b.

We next draw with center O and radius = 7.0 in. (equal to the pitch radius plus the addendum) the addendum circle.

The dedendum circle is drawn with radius of 4.84 in. (pitch radius minus the dedendum).

The outline of the tooth between the base circle and the dedendum circle is made a radial line.

The corner formed by the radial line and the dedendum circle is a source of weakness, which is relieved by rounding out with a small fillet curve of radius equal to the clearance.

The remaining teeth are, of course, duplicates and may be drawn by cutting a template of outline equal to the tooth originally drawn, or by transfer with the French or irregular curve.

	$14\frac{1}{2}°$ *Full-depth Involute*	$14\frac{1}{2}°$ *Composite*	20° *Stub Involute*	20° *Full-depth Involute*
Pressure angle, deg.	$14\frac{1}{2}$	$14\frac{1}{2}$	20	20
Addendum	$\frac{1}{\text{D.P.}}$	$\frac{1}{\text{D.P.}}$	$\frac{0.80}{\text{D.P.}}$	$\frac{1}{\text{D.P.}}$
Dedendum	$\frac{1.157}{\text{D.P.}}$	$\frac{1.157}{\text{D.P.}}$	$\frac{1}{\text{D.P.}}$	$\frac{1.157}{\text{D.P.}}$
Clearance	$\frac{0.157}{\text{D.P.}}$	$\frac{0.157}{\text{D.P.}}$	$\frac{0.20}{\text{D.P.}}$	$\frac{0.157}{\text{D.P.}}$
Tooth thickness*	$\frac{\text{C.P.}}{2}$	$\frac{\text{C.P.}}{2}$	$\frac{\text{C.P.}}{2}$	$\frac{\text{C.P.}}{2}$

* *Note:* No backlash has been provided in the tooth thickness proportion.

The $14\frac{1}{2}°$ full-depth and 20° stub-tooth system discussed above are most widely used. Two other systems of spur-gear tooth forms have been approved and are also designated as American Standards. These are the $14\frac{1}{2}°$ composite and the 20° full-depth involute systems. The teeth of the $14\frac{1}{2}°$ composite, $14\frac{1}{2}°$ full-depth, and the 20° full-depth system are all of the same proportions. The table on page 169 gives the comparison of tooth-form proportions for the four standard systems.

The $14\frac{1}{2}°$ *composite* system is used primarily when teeth are cut by the milling-cutter method. In the standard gear systems, the teeth are made conjugate to a rack of the system which becomes the basis of reference, or the "basic rack." The basic rack for the $14\frac{1}{2}°$ composite system is shown

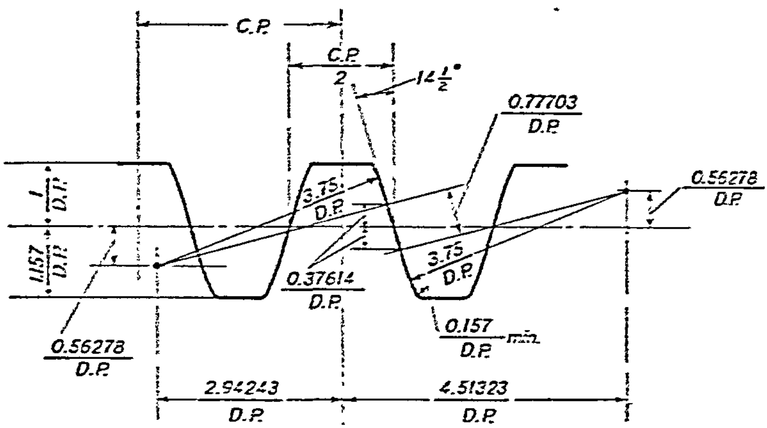


Fig. 192

in approximate form in Fig. 192. This system has been developed from the cycloidal tooth forms of the earlier gears, and the circular arcs which form portions of the face and flank have been selected so that they very closely approximate cycloidal curves. The straight-line portion of the tooth outline is an involute section. This modification of a completely involute rack tooth form retains most of the advantages of the involute system; it also tends to avoid interference and subsequent relief by undercutting in the case of small pinions.

The remaining systems are generally used when tooth-generating methods are employed for cutting. The $14\frac{1}{2}°$ *full-depth* system is most frequently used when the number of teeth is relatively large; with low tooth numbers the 20° *full-depth* system is most satisfactory. The 20° *stub-tooth* system is used primarily in automotive transmissions, and in heavy mill gearing where the greater strength of the short teeth is of advantage.

72. Normal Pitch. The normal pitch is the length of arc of the base circle from a point (e.g., the origin of the involute) on one tooth to the corresponding point on the next tooth. It is therefore equal to the circumference of the base circle divided by the number of teeth, or

$$\text{N.P.} = \frac{\pi D}{T}$$

in which

$$\begin{aligned} \text{N.P.} &= \text{Normal pitch} \\ D &= \text{Diameter of base circle} \\ T &= \text{Number of teeth.} \end{aligned}$$

Let us, once again, return to the unwinding of a taut cord as a description of the nature of an involute. Then successive points of the cord, like its end, are describing involutes of the base circle, and all these involutes are identical in that while the end point goes further than other points, its path is the same as theirs, and the successive portions of the cord are differing only in that they generate smaller portions of the same basic involute.

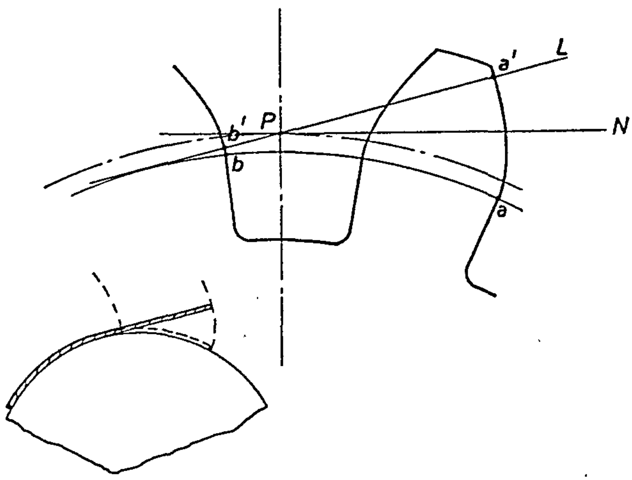


Fig. 193

This is illustrated in Fig. 193. The normal pitch has been defined as arc ab of the base circle. As the cord unwinds points a and b remain at constant distance from each other. When the cord lies on the pressure line we find that the line $a'b'$, now a straight line, is equal to the normal pitch. Then the normal pitch is equal to the distance from a point on the surface of one tooth to the point on the corresponding surface of the next tooth, measured along the pressure line.

73. Relationships in Involute Gearing. The condition, basic in involute gears, that the path of contact is a straight line reduces the number of variables encountered in the geometry of the involute system as compared with other systems of gearing.

These relationships, in general, may be summarized by pointing to the trigonometric function "$\cos\theta$" as their common basis. θ is the pressure angle.

For example, upon consulting Fig. 188 we note that

$$\frac{da}{dP} = \cos\theta$$

or

$$\frac{\text{Radius of base circle}}{\text{Radius of pitch circle}} = \cos\theta$$

and

$$\frac{D_{B.C.}}{D_{P.C.}} = \cos\theta$$

where $D_{B.C.}$ = Diameter of base circle

and $D_{P.C.}$ = Diameter of pitch circle

Then

$$\frac{\dfrac{\pi D_{B.C.}}{T}}{\dfrac{\pi D_{P.C.}}{T}} = \cos\theta$$

or, interpreting,

$$\frac{\text{Normal pitch}}{\text{Circular pitch}} = \cos\theta.$$

The relationship between path of contact and arc of action is another in which the ratio is the basic "$\cos\theta$."

We can prove that this is true by turning to Fig. 194. Here two teeth, one on each gear, are shown in contact at point a. Contact is just starting, for the addendum of the follower's tooth cuts the pressure line PL at a. By definition, line aP is the path of approach. The dotted outlines of this pair of teeth indicate their position when they are in contact at the pitch point. A line fP may be used to measure the angle of approach. This same line is in position fb just at the instant contact is starting, and angle α is the angle of approach of the follower F.

Then arc bP is, by definition, the arc of approach.

Now let us recall the unwinding cord as the analogy of our involute. If we imagine aT to be the cord, and wrap it back upon the base circle of the follower, its end point a follows the tooth outline to point s.

At the pitch point we may repeat this observation. Following along the tooth outline from P to the base circle by an identical involute path we arrive at point u. Then the line aP is equal in length to arc su.

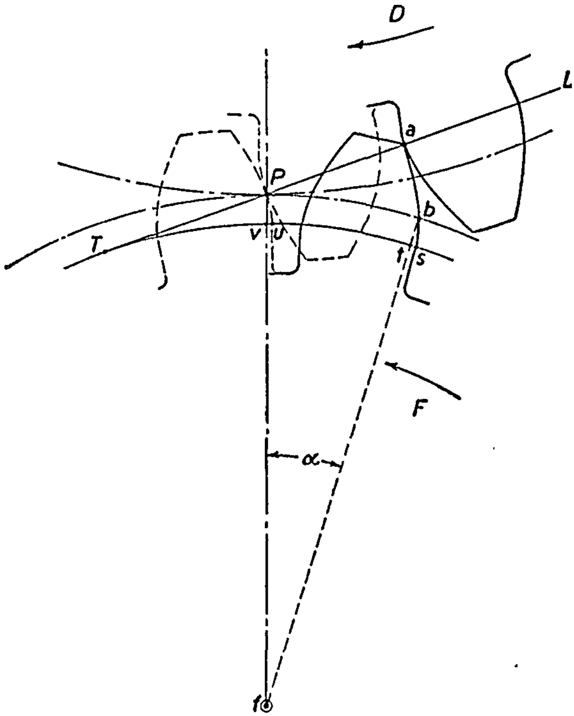


FIG. 194

But $$st = uv.$$

Then $$\text{arc } su = \text{arc } tv.$$

Now arc tv is an arc of the base circle, and

$$\frac{tv}{bP} = \frac{\text{Radius of base circle}}{\text{Radius of pitch circle}} = \cos\theta.$$

But $$aP = tv.$$

Then $$\frac{aP}{bP} = \cos\theta.$$

Interpreting, the ratio of the path of approach to the arc of approach equals $\cos\theta$.

The ratio of the path of recess to the arc of recess will also equal $\cos\theta$.

Or, summing up the component parts, the ratio of the path of contact to the arc of action equals $\cos\theta$.

The minimum permissible arc of action was shown in Art. 65 to be the circular pitch. Then with

Min. arc of action = C.P.

Min. arc of action $\times \cos\theta$ = C.P. $\times \cos\theta$; or, *minimum permissible path of contact = Normal Pitch.*

74. Limiting Conditions in Involute Gears. Thus far we have called attention to but one limiting condition which must be placed upon the design. The minimum value of arc of action in all gears has been found to be the circular pitch, and an equivalent demand in involute gears is that the minimum path of contact is the normal pitch.

Let us now observe the necessity for other limiting conditions affecting the tooth profile.

If the addendum of one or both gears is increased, a longer path of action results. This is desirable, since more pairs of teeth are brought into contact at the same time, resulting in improved distribution of the total load and smoother action. There is present, however, an accompanying danger of interference with the fundamental demand for conjugate action.

If we recall that an involute starts at the base circle we shall realize that there can be no conjugate action between the involutes beyond that point.

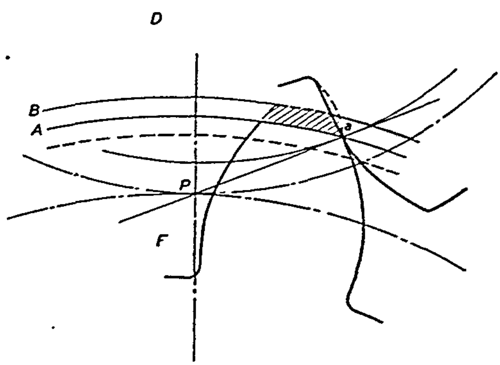


Fig. 195

Figure 195 illustrates the effect produced by an addendum (in this case, of the follower) which is too large.

Since the tooth profile is involute to the base circle only, point a, the point of tangency between pressure line and base circle of the driver, marks the limit of conjugate action on the involutes. If, therefore, any addendum circle of the follower be drawn whose radius does not carry it beyond point a (and beyond here means further from the pitch point P) the contacting action is confined to the region where conjugate action is provided for. The dotted addendum circle of Fig. 195 lies within point a and is therefore a

satisfactory addendum circle. The addendum circle marked A passes through the point of tangency. It is, in this case, the largest addendum circle which is satisfactory.

That conjugate action is disturbed when the addendum passes outside the point of tangency may be shown by the example illustrated in Fig. 196.

Here are found an involute pinion and rack in contact at point a, which is the point of tangency of the base circle of the pinion. The addendum of the rack passes through point a.

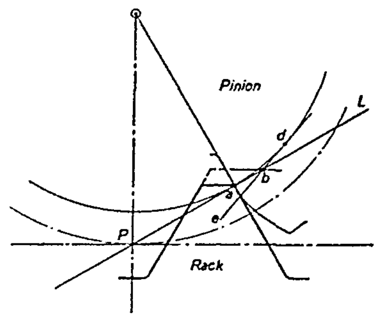


FIG. 196

All contact between the involutes must take place along the pressure line PL, which is of constant inclination. Point a lies on this line, and the involutes of both pinion and rack have their common normal along the pressure line shown.

The pressure line, in involute conjugate action, is always tangent to the base circle and always passes through the point of contact, serving as the common normal of the tooth curves and satisfying the fundamental law of gearing.

If a greater addendum is chosen, as shown by the dotted outline on the rack tooth, and the teeth remain otherwise undisturbed, the first point of contact must lie at point b, where the new rack addendum first cuts the constant-inclination pressure line. But the pressure line must remain tangent to the base circle of the pinion to be normal to its involute. Only two lines may pass through point b and be tangent to the base circle. These are shown as bd and ab. ab cannot be normal to the given involute of the pinion, for the involute will then be in the wrong direction to form the face of tooth. Then bd is the only line which can satisfy the demand for conjugate action, as far as the pinion involute is concerned. However, this line is no longer normal to the involute of the rack tooth, and the fundamental law cannot be satisfied, with the attempted addendum.

The extreme point of contact on the given pressure line which *will* allow a pressure line to be normal to both involutes and tangent to the base circle is point a. Then the maximum addendum circle of any gear is that which passes through a, the point of tangency of the mating gear.

If, in the search for longer paths of contact, we attempt such an addendum circle as the one marked addendum circle B of Fig. 195, we shall find, as the figure indicates, that the metal of the follower's tooth is overlapping the metal of the driver's tooth. Such an addendum circle has the effect of attempting to cause two bodies to occupy the same space at the same time—

A line dP establishes pitch point P, through which the pressure line PL is drawn making an angle of 20° with the normal to dP.

The addendum circle is next drawn, with addendum circle radius equal to 6.4 inches.

We are now ready to search for the smallest pinion.

If any line, say $b'f'$, be drawn perpendicular to the pressure line it will meet the line of centers at f', and b' will be the point of tangency of a base circle whose center is f', and Pf' will be the pitch radius of this pinion. This pinion is an unsatisfactory one, for the addendum circle of the given gear cuts the pressure line beyond b', the point of tangency, and interference results. Moving the point of tangency outward—away from the pitch point as, for example, to b^3—increases the pitch radius of the pinion. When point b^2 is used as point of tangency and line b^2f^2 erected perpendicular to the pressure line, we establish Pf^2 as pitch radius of a pinion. This is a satisfactory arrangement, for the addendum circle of the gear does not cut the pressure line beyond the point of tangency of the pinion, and there is no interference. Pf^2 is then the desired pitch radius of the smallest possible pinion. Pf^2 may be measured, if the graphical solution is to be relied upon and precise drafting has been employed.

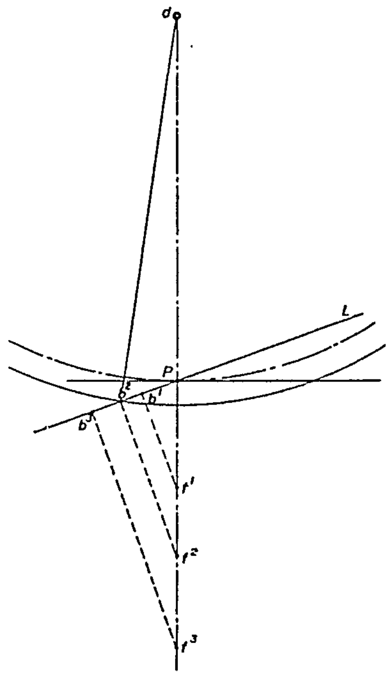


FIG. 197

If the drawing has been used simply as a sketch to clarify the conditions, and serve as a basis for calculations—and this method is, in general, the most satisfactory one for such problems as the present one—we may calculate the pitch radius, Pf^2.

$$\text{Angle } dPb^2 = 90° + 20° = 110°$$

$$db^2 = 6.4 \text{ in.}$$

In triangle dPb^2

$$\sin db^2P = \frac{6 \times \sin 110°}{6.4} = 0.881$$

$$\text{Angle } db^2P = 61.8°$$

$$\text{Angle } b^2dP = 180 - (110 + 61.8) = 8.2°$$

$$Pb^2 = 6 \times \frac{\sin\ 8.2°}{\sin\ 61.8°} = 0.971 \text{ in.}$$

$$Pf^2 = \frac{0.971}{\sin\ 20°} = 2.84 \text{ in.}$$

Then the pitch diameter of the smallest pinion is $2 \times 2.84 = 5.68$ in. With diametral pitch $= 2$, the number of teeth on the pinion is, theoretically, equal to 11.36.

Since the number of teeth must be an integer, we move to the next larger integer which is 12. This is a change which causes the pitch radius to increase toward Pf^3, and the point of tangency to move outward toward b^3. Since the addendum of the large gear is remaining fixed, this addendum circle will now be cutting the pressure line inside the point of tangency of the pinion, so that no interference ensues.

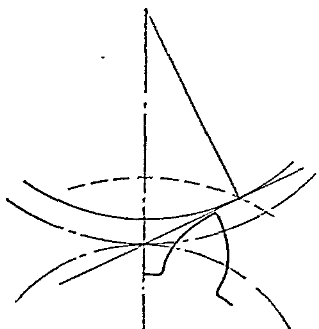


FIG. 198

The solution of 12 teeth gives the smallest possible pinion to run with the given gear.

There is another limitation affecting the position of the addendum circle. If, under the permissible conditions, a gear is of such dimensions as those shown in Fig. 198 it is possible that the involutes forming the two sides of a tooth will meet before any limit of interference is reached. The addendum circle now lies at the junction, or point of the tooth. When load is imposed the tooth point becomes the weakest section and will break. In this case metal must be removed, and the addendum circle's diameter diminished until sufficient cross-sectional area is established to present a proper tooth strength. The maximum path of contact is now limited by tooth strength rather than by interference possibilities.

PROBLEMS

221. Two parallel shafts 18 in. apart are connected by a pair of involute gears in external contact. The speed ratio is 2 : 1, the diametral pitch = 1, pressure angle = $14\frac{1}{2}°$. Draw the pitch, addendum, dedendum, and base circles. Design two teeth on each gear, having one pair in contact at the pitch point. Use tooth dimensions given in American Standard table.

222. Solve Problem 221, changing only the pressure angle which is to be made 20°. Use 20° stub involute proportions.

223. The pitch diameter of an involute gear is 12 in., when meshed with another gear. The diametral pitch = 1, pressure angle = 20°, dimensions of teeth standard. Draw a complete tooth of the gear, and report the addendum distance, the dedendum distance, and the circular pitch. Use 20° stub involute proportions.

224. Two parallel shafts whose centers are 6 in. apart are to be connected by a pair of involute gears so that they rotate in the same direction. The speed ratio is to be 1 : 2, pressure angle = 20°, and diametral pitch = 2. Determine the length of the paths of approach, recess, and contact; the angle of action; the arc of contact; the ratio of the arc of contact to the circular pitch. Use (a) 20° stub involute proportions; (b) 20° full-depth involute proportions.

225. Design a 20-tooth involute gear *A* in external contact with a 30-tooth gear *B*. Diametral pitch = 2. Pressure angle = 20°. Tooth dimensions standard stub involute. Draw three teeth on each gear, having one pair in contact at the pitch point.

Report in a table the following values for the pair of gears:

(a) Pitch diameters
(b) Addendum diameters
(c) Circular pitch
(d) Normal pitch
(e) Ratio of path of contact to normal pitch
(f) Number of pairs of teeth in contact
(g) Arc of contact

For each gear, also report

(h) The angles of approach, recess, and action.

226. Solve Problem 225, modifying the data as follows:

Diametral pitch = 1. Gear *A* has 12 teeth. Gear *B* has 14 teeth.

227. Solve Problem 225, modifying the data as follows: Gear *A* has 24 teeth. Gear *B* has 48 teeth. Diametral pitch = 2. Pressure angle = $14\frac{1}{2}$°.

228. Solve Problem 225, modifying the data as follows:

Diametral pitch = 2. Contact is to be internal. Pinion *A* has 14 teeth, and speed ratio is 1 : 3, with the pinion acting as driver.

229. Determine the least pressure angle which may be used in designing duplicate involute gears of 12 teeth each, to run together. Use diametral pitch = 1. *Ans.* 14.7°.

230. Determine the least pressure angle which may be used in designing duplicate involute gears of 18 teeth each, to run together.

231. What is the smallest number of teeth that may be used on a pair of equal involute gears if the pressure angle is 20°, and the diametral pitch = 1. The addendum distance is to be standard (stub involute).

232. Solve Problem 231 with data modified so that the pressure angle = $14\frac{1}{2}$°.

233. Determine the ratio of the arc of action to the circular pitch in Problem 232.

234. Two equal involute gears of 30 teeth have a pressure angle of $14\frac{1}{2}$°, diametral pitch = 3. The arc of contact is to be made equal to twice the circular pitch. Determine the necessary addendum circle diameter. Will there be interference?

235. Using the data of Problem 234, determine the maximum permissible addendum distance for the two gears.

236. Using the standard addendum value, determine the smallest pinion, diametral pitch = 2, which may be used to drive a rack with pressure angle equal to (a) $14\frac{1}{2}$°, (b) 20°.

237. Determine the number of teeth on the smallest involute pinion which may be used with a 64-tooth gear. Diametral pitch = 5. Pressure angle = 20°. Dimensions standard (stub involute). Pressure angle is to remain constant. *Ans.* 13.

238. Determine the number of teeth on the smallest involute pinion which can be used with a 24-tooth gear. Diametral pitch = 2. Pressure angle = $14\frac{1}{2}°$. Dimensions standard. Pressure angle is to remain constant.

239. Determine the maximum speed ratio which is obtainable using a pair of involute gears of diametral pitch = 3. The driver has 36 teeth. The pressure angle is 20° and remains constant. All dimensions are standard.

240. Determine whether a 48-tooth involute gear may be used to drive a 12-tooth gear. Diametral pitch = 2. Pressure angle = $14\frac{1}{2}°$. All dimensions standard.

241. A 64-tooth involute annular is to drive a 24-tooth pinion. Pressure angle = 20°. Diametral pitch = 4. All dimensions standard. Determine whether or not this arrangement is satisfactory.

242. A pair of involute gears is to be designed with speed ratio = 1 : 4. Pressure angle = 20°. The pinion is the driver. The arc of approach must be equal to the circular pitch. Determine the least number of teeth on each gear, with diametral pitch = 2.

75. Change in Line of Centers of Involute Gears. An involute gear has certain properties which are inherent and fixed, while other properties are dependent upon the conditions under which the gear is used in practice.

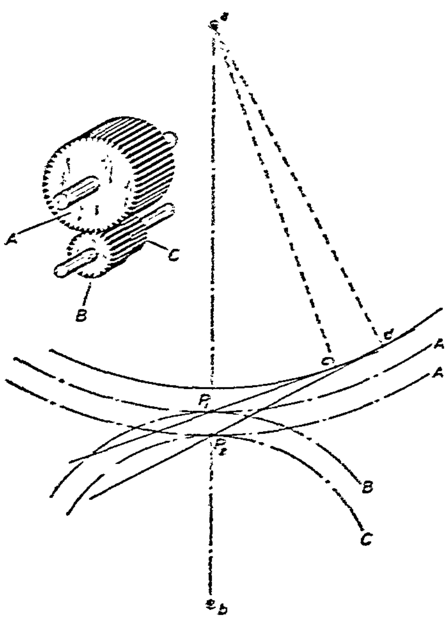


FIG. 199

For example, the properties of tooth outline are cut into the metal and, once established, cannot be changed without further cutting. The addendum circle is the outermost circle of the gear, and of course could not be changed without physically modifying the gear. The dedendum circle is similarly in permanent position once the gear has been manufactured.

The base circle, established as the origin of the involute curves, is also a fixed property of a given gear. The number of teeth remains unalterable, and so the normal pitch is a fixed distance.

The pitch circle, however, is a different kind of property. No gear has a pitch circle until it mates with another gear. While addendum, dedendum, and base circles are physically present and fixed, and so may be located by inspection of the gear, the pitch circle has served only as a background for discussion of action, and to represent the rolling cylinders which have evolved into toothed wheels. The pitch circle cannot be found by inspection of one gear, but takes its position when two gears are brought into

mesh. Its location is dependent upon the properties of both mating gears, rather than of one gear. Within limits which we shall discuss, any involute gear may be made to run with one or more other involute gears. In each case, the location of the pitch circle will be governed by the speed ratio of the pairs of mating gears.

For example, in Fig. 199 we find a large gear A driving gears B and C at the same time, with B and C mounted on the same axis, but free to turn relative to each other.

Then A must have, at the same time, two pitch circles.

In the case shown

Gear A has 48 teeth
" B " 24 "
" C " 20 "

The line of centers ab = 18 inches.

Then, when we analyze the action of A and B, assuming that gear A is the driver and B is follower, we find that

$$\text{Speed ratio} = \frac{\omega_F}{\omega_D} = \frac{T_A}{T_B}$$

where T_A is the number of teeth on $A = 48$
and T_B " " " " " " $B = 24$

Then

$$\frac{\omega_F}{\omega_D} = \frac{48}{24} = \frac{2}{1}.$$

But from our concept of the pitch circles as rolling cylinders,

$$\frac{\omega_F}{\omega_D} = \frac{D_A}{D_B}$$

where D_A is the pitch diameter of A
and D_B " " " " " B

$$\frac{D_A}{D_B} = \frac{2}{1}$$

$$\frac{D_A}{2} + \frac{D_B}{2} = 18 \text{ in.}$$

From these simultaneous conditions, we find that

$$D_A = 24 \text{ in.}$$
$$D_B = 12 \text{ in.}$$

Pitch circles of 24-inch diameter on A and 12-inch diameter on B are the background equivalent rolling cylinders. These are represented by the pitch circles drawn to meet at pitch point P_1.

If gears A and C are analyzed we find that in this case speed ratio

$$\frac{\omega_F}{\omega_D} = \frac{T_A}{T_C} = \frac{48}{20} = \frac{12}{5}$$

$$\frac{D_A}{D_C} = \frac{12}{5}.$$

But

$$\frac{D_A}{2} + \frac{D_C}{2} = 18 \text{ in.}$$

Then $$D_A = 25.4 \text{ in.}$$
and $$D_C = 10.6 \text{ in.}$$

The pitch circles for A and C are represented by the circles drawn to meet at their pitch point P_2.

It now becomes evident that, as in the case of gear A, a gear may have more than one pitch circle, depending upon the conditions of pairing it with other gears.

We have built up a series of definitions of gearing properties from the basis of the pitch circle. If the pitch circle is subject to change depending on its mating action with other gears, these properties will change to accompany the pitch circle.

Returning to the illustrative example we have been using to illustrate the changing nature of the pitch circle, let us observe the differences which arise between the drive of gears A and B, and that of A and C.

$$\text{Diametral pitch} = \frac{T}{D}.$$

For gears A and B the diametral pitch is

$$\frac{T_A}{D_A} = \frac{48}{24} = 2$$

or

$$\frac{T_B}{D_B} = \frac{24}{12} = 2.$$

For gears A and C the diametral pitch is

$$\frac{T_A}{D_A} = \frac{48}{25.4} = 1.89$$

or

$$\frac{T_C}{D_C} = \frac{20}{10.6} = 1.89$$

It becomes evident, from this example that the diametral pitch of a gear may change and is dependent, like the pitch circle, upon its operation with another gear to establish its value.

Other properties will have values dependent upon pairing of gears, and not upon the physical form of the single gear. In fact, all of the properties which rest upon the pitch circle as a basis of definition will change with the pitch circle, and have no significance until, by pairing gears, the pitch circle is definitely located. These properties include the circular pitch, addendum, dedendum, working depth, pressure angle, etc.

For example, the pressure line, when gear A and B of Fig. 199 are considered, is tangent to the base circle of gear A at point c, and passes through their pitch point P_1.

When gears A and C are considered, the pressure line is dP_2, for it must remain tangent to the base circle of A and pass through the pitch point of A and C.

The base circle of an involute gear is physically located on the gear, and is established when the gear is cut. It remains fixed in location regardless of the mating gear, and a property like normal pitch, which is dependent upon the base circle for definition, remains a permanent quantity regardless of the mating of this gear with any other.

The principles underlying the ability of one involute gear, like A, of Fig. 199 to mate with more than one other gear, like B and C, without disturbing the conjugate action find another opportunity of application in the case of two gears whose line of centers is to be changed in length.

Our analogy of the action of base circles and path of contact, and that of pulleys with crossed belt can be useful in clarifying the condition of fixity of the base circles and flexibility of the pitch circles.

In Fig. 189 we note two pulleys with crossed belt. The speed ratio

$$\frac{\omega_F}{\omega_D} = \frac{D_D}{D_F}$$

depends upon the diameters of the pulleys, and is not influenced by changing the length of the line of centers.

The analogous action of the base circles and pressure line of Fig. 189 is, likewise, dependent upon the diameters of the base circles or the equivalent interpretation into number of teeth and does not depend upon the length of the line of centers. The conjugate action of involutes of these base circles fulfills the fundamental law of tooth profile, for in our observations of Art. 70, we found that the action of these involutes is independent of the length of the line of centers.

As the line of centers changes in length, the pressure line, remaining ever tangent to the base circles, will change in inclination, with resulting change in pressure angle.

Let us investigate the relationships between properties deriving from the pitch circle concept, and those which remain fixed because they are dependent only upon the base circle.

Gears A and B, of Fig. 200, are given.

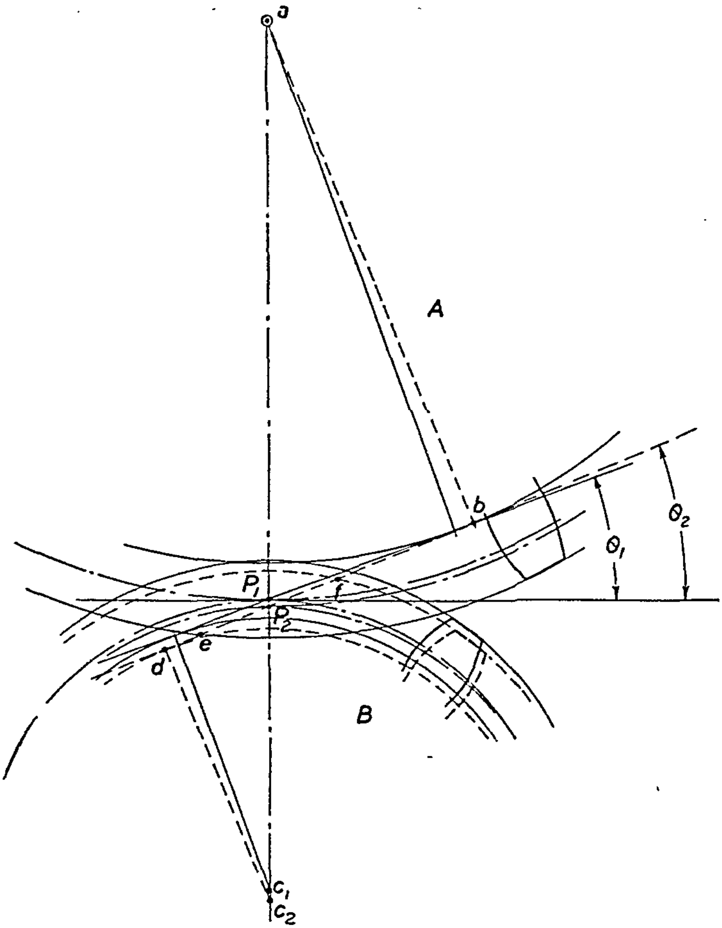


FIG. 200

These were designed to have the following properties when running together, with pitch point at P_1.

	Gear A	*Gear B*
No. of teeth	24	12
Diametral pitch	2	2
Pressure angle	20°	20°
Addendum	0.4 in.	0.4 in.

Then the following values may be obtained from the given data.

	A	*B*
Pitch circle diameter	12.00 in.	6.00 in.
Circular pitch	1.57 in.	1.57 in.
Normal pitch	1.47 in.	1.47 in.
Path of approach	0.87 in.	0.87 in.
Path of recess	0.98 in.	0.98 in.
Path of contact	1.85 in.	1.85 in.
Line of centers	9.00 in.	9.00 in.
Diameter of addendum circle	12.80 in.	6.80 in.

Now let us move the gears apart until they have a line of centers equal to 9.10 in., as shown in Fig. 200. The change will be effected by keeping *A* at its original position, and moving *B* downward, from center c_1 to center c_2.

The same gears have been employed. We have, then, the following properties unchanged.

	A	*B*
No. of teeth	24	12
Normal pitch	1.47 in.	1.47 in.
Diameter of addendum circle	12.8 in.	6.8 in.

The new location of the centers has caused the changes which are outlined below.

We should first observe the effect upon the establishment of the pitch circles, when mating these two gears with line of centers equal to 9.10 in.

Since the contact between the gears is made through the teeth, the speed ratio is fixed by the numbers of teeth.

Then

$$\frac{\omega_B}{\omega_A} = \frac{24}{12} = \frac{2}{1}$$

and this ratio remains unaltered by the change in length of line of centers.

The pitch circles of this pair must satisfy the concept of rolling cylinders which would yield an equivalent speed ratio.

Then

$$\frac{2}{1} = \frac{D_A}{D_B}$$

where D_A is the pitch diameter of gear *A*
and D_B " " " " " " *B*

But
$$\frac{D_A}{2} + \frac{D_B}{2} = 9.10$$

With these two simultaneous equations, we find that

$$D_A = 12.14 \text{ in.}$$

and

$$D_B = 6.06 \text{ in.}$$

These pitch circles are tangent at pitch point P_2.

The pressure line must pass through the pitch point. We can check this by noting that in similar triangles abP_2 and c_2dP_2

$$\frac{ab}{c_2d} = \frac{aP_2}{c_2P_2}$$

for line bP_2d is the continuous pressure line, and ab and c_2d are perpendicular to it. ab and c_2d are radii of the base circles. They were originally, and will continue to be, in the ratio of 2 : 1. Then the new pitch radii, aP_2 and c_2P_2, as corresponding sides of similar triangles, maintain the same 2 : 1 ratio.

The pressure angle has changed in inclination. This line remains tangent to the base circles and passes through the new pitch point. The value of pressure angle under the new conditions is

$$\theta_2 = \cos^{-1}\left[\frac{ab}{aP_2} = \frac{5.64}{6.06} = 0.931\right] = 21.5°$$

The addendum circle of gear A has not moved. Then the path of recess is now $P_2e = 0.78$ in., which is somewhat shorter than it was before the line of centers changed.

The addendum circle of gear B has been displaced, as far as its absolute position is concerned, but its position relative to the axis of gear B remains unchanged. This addendum circle cuts the pressure line at point f, and $P_2f = 0.78$ in. is the path of approach.

These new path values suggest the limit which we must impose. The gears may be separated, and the line of centers increased, until the path of contact—which is diminishing during the separation—becomes equal to the normal pitch. Further separation would violate the law that "minimum permissible path of contact is the normal pitch." The new path of contact, $fe = 1.56$ in., which is satisfactory.

The possibilities of separation of involute gears are not of theoretical interest alone. They form one of the advantages which has established the involute system so successfully. A change in the line of centers will occur in practice when the loading of the gears causes deflection of the supporting shafts, or when wear of the bearing surfaces disturbs the original alignment. The ability of such gears to maintain conjugate action despite such disturbances is a definite advantage over systems which depend upon the permanence of the conditions of the original design.

We may also make use of the principle of separation resulting in changed

pitch circles, but not affecting the conjugate action, in other conditions of mating gears.

Two or more gears of different numbers of teeth may be mounted on the same axis and be driven by one driving gear. Such a case was shown in Fig. 199. Or again, two or more racks originally designed with different obliquity or pressure angle may be driven satisfactorily by one driving gear.

This is an interesting development of the underlying principle, and we shall examine such an application. Two racks and driving pinion are shown in Fig. 201.

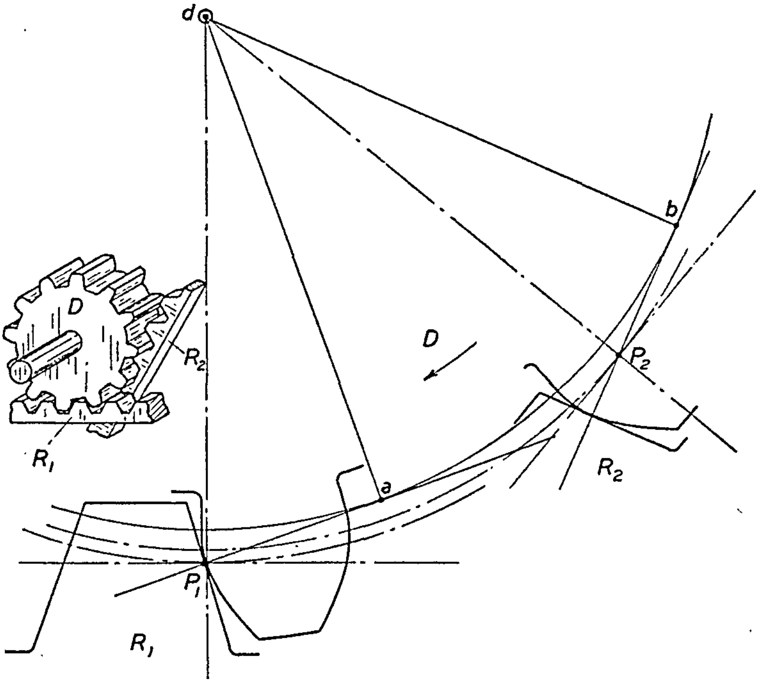


Fig. 201

Rack R_1 was designed with pressure angle of 20°, and rack R_2 with pressure angle of 14.5°. Then the teeth of R_1 make an angle of 20° with the line of centers, and those of R_2 an angle of 14.5°.

The combination will operate with proper conjugate action, for two pitch circles of the pinion D are established as background. In the case of the drive D and R_1 the pitch point will lie at P_1, the pressure line at P_1 being tangent to the base circle of the pinion (at a) and normal to the face of the tooth. We must recall that the normal to the tooth surface must pass through the pitch point.

The drive D and R_2 will have pitch point at P_2, in order that the pressure line be tangent to the base circle (at b) and normal to the face of the teeth of rack R_2.

The linear velocities of the two racks will differ. The linear velocity of rack $R_1 = \omega_D \times dP_1$, and that of $R_2 = \omega_D \times dP_2$.

PROBLEMS

243. Two equal standard involute $14\frac{1}{2}°$ gears of 28 teeth, diametral pitch = 2, are to be mounted with distance between centers equal to 14.5 in. Determine whether or not the center to center distance is permissible.

244. Using the gears of Problem 235, determine the maximum center to center distance with which the gears may be satisfactorily mounted.

245. Two 20° full-depth involute standard gears of 18 and 14 teeth, designed with diametral pitch = 1, are to be used in a machine drive, which requires that their axes be mounted 16.4 in. apart.

Determine the values of:

(a) Pitch diameters
(b) Pressure angle
(c) Circular pitch
(d) Normal pitch
(e) Path of contact.

246. Solve Problem 245, except that a standard $14\frac{1}{2}°$ gear of 32 teeth, diametral pitch = 2, is to be used with a gear of 26 teeth, with axes 14.8 in. apart.

247. Two involute gears, 16 teeth each, diametral pitch = 1, pressure angle 20°, are to run together. Draw two teeth on each, having one pair in contact at the pitch point. All dimensions are standard (full-depth involute).

On the same shaft with one gear a 14-tooth gear is to be mounted, meshing with the other 16-tooth gear. Draw two teeth of the 14-tooth gear, in contact with the 16-tooth gear. The addendum and dedendum diameters are to be the same as for the first pair.

Determine the path of contact and the ratio of the path of contact to the normal pitch for each pair.

248. An additional gear, C, is to be mounted on the same shaft as gear B of Problem 225, so that it will mesh with A. C is to have 32 teeth, and its addendum and dedendum diameters are to be the same as those of B.

Report in a table the values called for in that problem, for the pair of gears A and C.

249. Design a drive in which a 16-tooth pinion A, designed as a $14\frac{1}{2}°$ standard gear, diametral pitch = 1, meshes with three racks, B, C, and D, placed 120° apart.

Rack B has a pressure angle = $14\frac{1}{2}°$.
Rack C has a pressure angle = 20°.
Rack D has a pressure angle = 30°.

Draw all teeth on A, and two teeth on each rack. The addendum of each rack is to be made the maximum permissible addendum.

Determine the linear velocity of each rack if the angular velocity of the pinion is 60 r.p.m.

Determine the ratio of path of contact to normal pitch for each rack.

76. Cycloidal Gear Teeth. Cycloidal curves were formerly widely used in tooth profiles, but when compared with the curves of the profiles of the involute system it will be found that they have marked disadvantages; the

cycloidal gears must be held, for example, in exactly the same position as their original alignment, or the conjugate action is disturbed. In addition, the varying pressure angle of the cycloidal system results in a varying force system which disturbs smooth action and encourages wear. A most important disadvantage, which has served to cause abandonment of the cycloidal

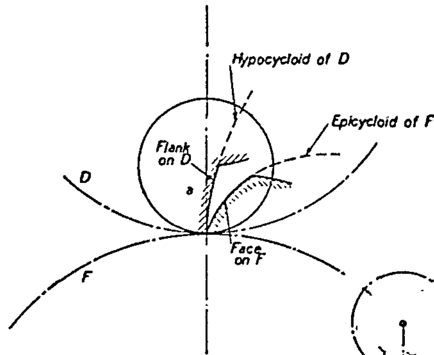
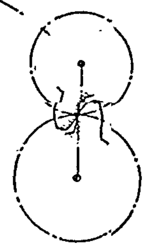


FIG. 202



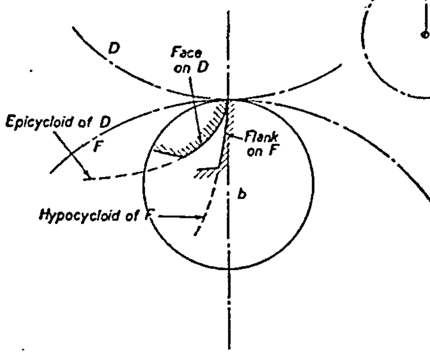


FIG. 203

system, lies in the greater practical difficulties of manufacturing gears with cycloidal tooth forms.

The form is used occasionally when the gears are to be cast, or when cycloidal "tips" are added to involute tooth profiles to increase the length of the path of contact, and we shall note the essential features of the geometric design.

The generation of the tooth profiles is shown in Fig. 202.

A circle, a, known as the describing circle is shown in position on upper and lower pitch circles in Fig. 202.

When the describing circle rolls internally with pure rolling contact on

the upper pitch circle, a point on its circumference will generate the hypocycloid shown. A portion of this hypocycloid becomes the flank of the tooth of the upper gear.

If now the same generating circle is allowed to roll externally upon the lower pitch circle it will generate an epicycloid, a portion of which becomes the face of the tooth of the lower gear.

Now another describing circle, *b*, shown in Fig. 203, is introduced to produce the face of the upper gear tooth, and the flank of the lower, which completes the tooth form.

The same describing circle must be used for face of tooth on one gear, and flank of tooth on the other. In practice, to secure a series of interchangeable gears, the same describing circle is used throughout the series.

Such gears satisfy the fundamental law of gearing, when the same describing circle is used to generate the face and flank which are in contact and as long as the pitch circles which served to direct the describing circle remain tangent.

77. Stepped Gears. We noted, while engaged in our study of the involute gear, the advantage of having more than one pair of teeth in contact for smoother action. With those tooth surfaces which were parallel to the axis, we can provide such overlapping of contact by increasing the height of the teeth or by using a greater diametral pitch, resulting in more teeth on each gear. Both of these methods result in some reduction in the strength of the tooth.

An improvement over the single spur gear is suggested in the stepped gear of Fig. 204. The load is here distributed over a series of steps so that

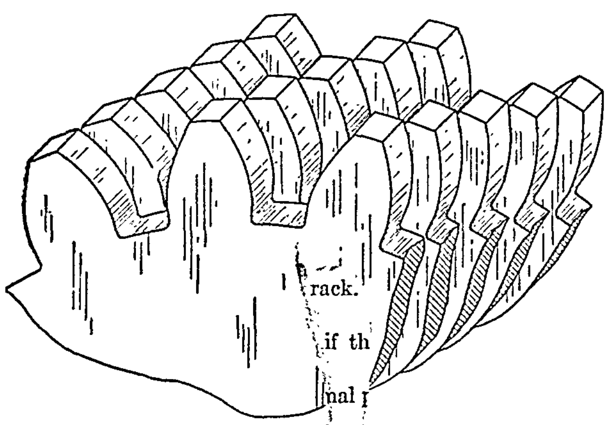


Fig. 204

smoother action results. One of the advantages lies in the fact that in distributing the total loading over the several steps, the average length of the

normals to the tooth profiles is reduced. With this reduction comes a reduction of the average amount of the velocity of sliding. In turn the work which must be spent in overcoming the friction due to the sliding of the teeth is reduced.

A stepped wheel consists of a number of equal wheels placed so that each wheel or step is set a little behind the preceding one. Now the number of teeth in contact at any time is increased in proportion to the number of steps. As we cut thinner slices of a given gear to form the steps we approach a gear made of a series of thin plate gears, each advanced slightly beyond its neighbor. The limit is shown in Fig. 205 where the center line of each tooth has become a helix traced on the pitch cylinder of the wheel.

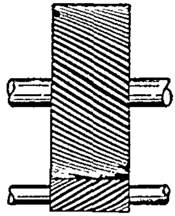


FIG. 205

78. Helical Gears are those evolved, as described in the preceding article, from stepped wheels when the laminated steps have been reduced in thickness to zero value and the resulting center line of each tooth is a helix.

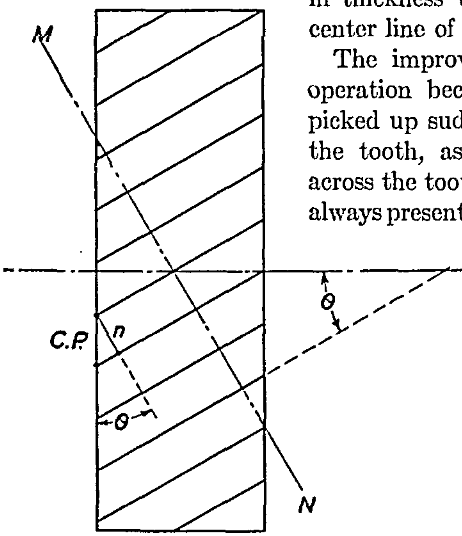


FIG. 206

The improvement with regard to smooth operation becomes evident. Contact is not picked up suddenly across the entire width of the tooth, as in spur gears, but progresses across the tooth, removing the shock tendency always present when loads are imposed suddenly in the engagement action of any driver and follower.

The angle which lies between a tangent to the helix at the surface of the pitch cylinder and an element of the pitch cylinder is called the *helix angle* θ. This angle is shown in Fig. 206.

The development of the surface of a helical gear is shown in the figure. The lines representing teeth are the imaginary center lines of such teeth at the level of the pitch cylinders. This simplification will make it possible to observe factors of relationship without the confusion of reading projected views of the entire tooth surfaces.

The line MN is drawn normal to the lines of the teeth. This line MN will become a helix if the development is wrapped around the pitch cylinder, and is called the *normal helix*.

The distance n is the distance between corresponding points on adjacent teeth. It is called the *Normal circular pitch*, and will be abbreviated N.C.P. As in the case of spur gears, the relation between normal circular pitch and normal diametral pitch, N.D.P., prevails.

$$\text{N.D.P.} = \frac{\pi}{\text{N.C.P.}}$$

In the figure we note that the normal circular pitch bears a definite relation to circular pitch, C.P. measured in the plane of rotation.

$$\text{C.P.} = \frac{n}{\cos\theta} = \frac{\text{N.C.P.}}{\cos\theta}$$

where θ is the helix angle.

The normal circular pitch of a mated pair of helical gears must be the same.

Since

$$\text{C.P.} = \frac{\text{N.C.P.}}{\cos\theta}$$

and the pitch circumference $\pi D = \text{C.P.} \times T$
where T is the number of teeth

$$\text{C.P.} \times T = \frac{\text{N.C.P.} \times T}{\cos\theta}$$

but the normal diametral pitch, $\text{N.D.P.} = \dfrac{\pi}{\text{N.C.P.}}$

Then

$$\text{C.P.} \times T = \frac{\pi \times T}{\text{N.D.P.} \times \cos\theta}$$

and

$$\pi D = \frac{\pi \times T}{\text{N.D.P.} \times \cos\theta}$$

from which

$$D = \frac{T}{\text{N.D.P.} \times \cos\theta}$$

which is the relationship between pitch diameter, number of teeth, normal diametral pitch, and helix angle.

Let us next consider relationships between a mated pair of gears, like the driver and follower whose developments are shown in Fig. 207. The helix angle of the driver is θ_D, and that of the follower is θ_F.

$$T_D = \frac{\pi D_D}{\text{C.P.}_D}$$

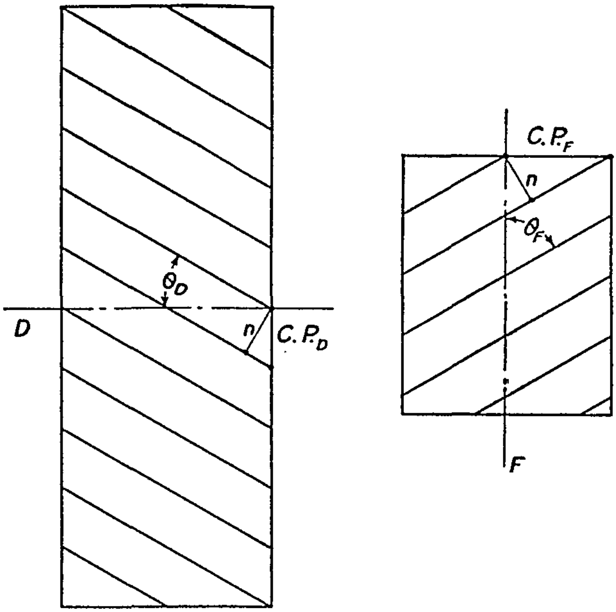


FIG. 207

where T_D is the number of teeth on the driver

D_D is the pitch diameter of the driver

$C.P._D$ is the circular pitch of the driver measured in the plane of motion. Similarly

$$T_F = \frac{\pi D_F}{C.P._F}$$

where T_F is the number of teeth on the follower

D_F is the pitch diameter of the follower

$C.P._F$ is the circular pitch of the follower measured in the plane of motion. Then

$$\frac{T_D}{T_F} = \frac{\dfrac{\pi D_D}{C.P._D}}{\dfrac{\pi D_F}{C.P._F}} = \frac{D_D \times C.P._F}{D_F \times C.P._D}$$

The normal circular pitch, N.C.P., must be the same for both gears.

$$C.P._D = \frac{N.C.P.}{\cos \theta_D}, \qquad \text{and} \qquad C.P._F = \frac{N.C.P.}{\cos \theta_F}$$

$$\frac{T_D}{T_F} = \frac{D_D \times \dfrac{N.C.P.}{\cos \theta_F}}{D_F \times \dfrac{N.C.P.}{\cos \theta_D}} = \frac{D_D \cos \theta_D}{D_F \cos \theta_F}.$$

Expressing this conclusion in words: *The ratio of the number of teeth on driver to the number of teeth on follower equals the ratio of the products of their respective pitch diameters and the cosines of their helix angle.*

The speed ratio of two helical gears, like that of spur gears, is inversely as the numbers of teeth,

or $$\frac{\omega_F}{\omega_D} = \frac{T_D}{T_F} \text{ in terms of numbers of teeth,}$$

or $$\frac{\omega_F}{\omega_D} = \frac{D_D \cos \theta_D}{D_F \cos \theta_F} \text{ in terms of pitch diameters and helix angles.}$$

A resolution of velocity vectors will bring out the velocity relationship most forcefully and afford opportunity for us to note particularly the component which is sliding velocity.

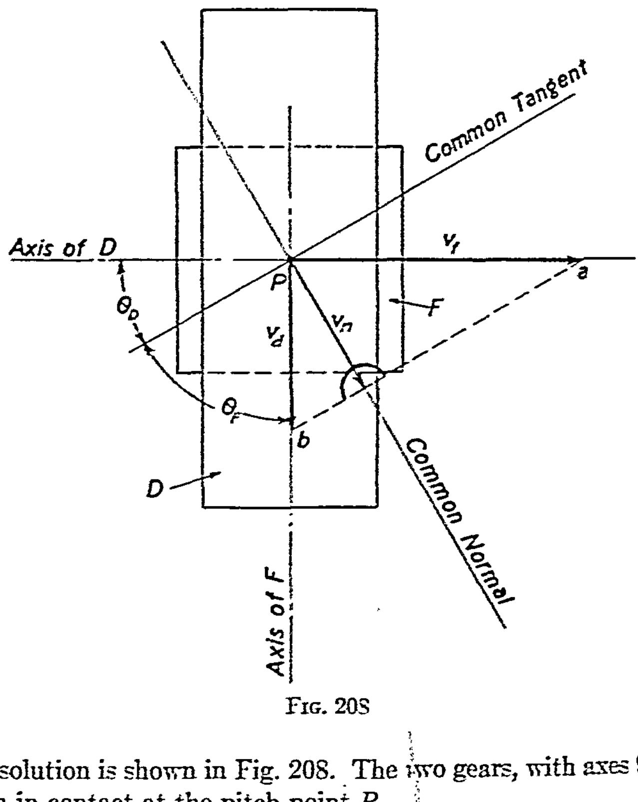


FIG. 208

This resolution is shown in Fig. 208. The two gears, with axes 90° apart, are shown in contact at the pitch point P.

The velocity of point P on D is called v_d and the velocity of point P on F is v_f.

In this sliding action, as in all sliding contact, v_n, the orthogonal component along the common normal, is an orthogonal component of both v_d and v_f. The rate of sliding has magnitude ab and inclination along the common tangent.

We can also note from this figure that the sum of the helix angles is equal to the angle between the shafts when helical gears are used to accomplish a drive between non-intersecting shafts, whether they are parallel or oblique to each other.

79. Herringbone Gears. When a pair of helical gears is in contact, the driving force on the tooth has a component parallel to the axis, or an end thrust. Such gears must therefore be mounted between properly designed thrust bearings.

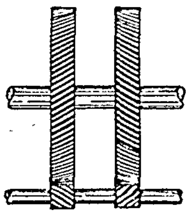
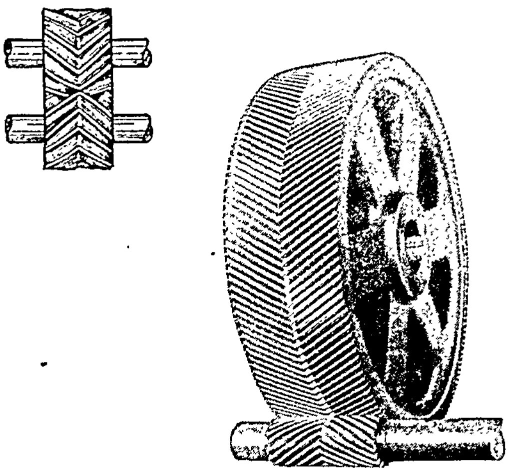


Fig. 209

This necessity may be avoided by placing two helical gears of the same helix angle but opposite directions of obliquity on the same shaft. An example is shown in Fig. 209. Now the end thrust of one gear is neutralized, as far as the supporting bearings are concerned, by an equal but opposite end thrust from the second gear on the same shaft.

This feature has led to the development of the *herringbone* type, a

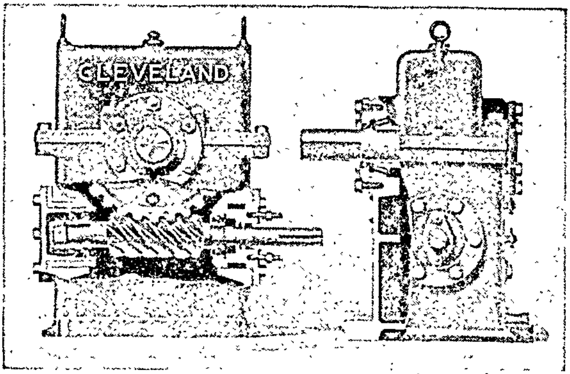


Courtesy the D. O. James Mfg. Co.

Fig. 210

single gear made in the form of two opposed helical gears. This type is shown in Fig. 210.

80. Worm Gearing. The *worm and wheel* shown in Fig. 211 form another example of helical gearing. The name is generally reserved for



Courtesy The Cleveland Worm & Gear Co.

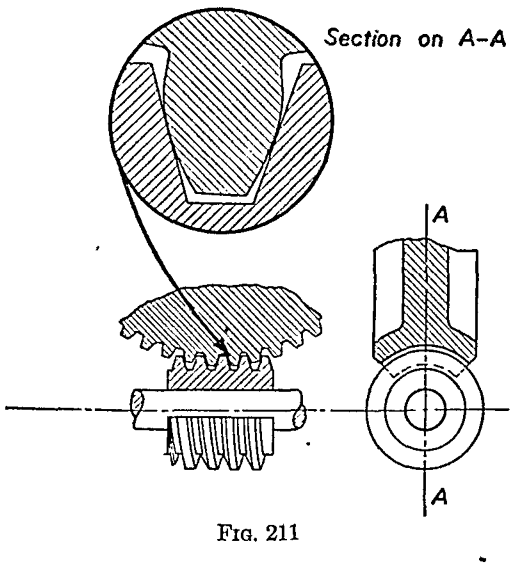


Fig. 211

cases where the axes are at right angles, and the outside surface of the wheel is no longer cylindrical but follows the curvature of the worm.

In most cases where high speed ratio is used, the worm and wheel afford a high-speed ratio, in compact form. Another advantage is the possibility of arranging the helix angles so that the drive will operate in one direction only. Such irreversible drives form, for example, a proper mechanism for automotive steering gears.

The action may be compared to that of a screw and nut, with the worm forming the screw and the wheel the nut. As in the case of nut and screw, the threads may be single or multiple.

Or we may note, as in Fig. 211, that if we inspect a section A–A through the axis of the worm, we find the same action as that of involute pinion and rack. If the worm is revolved, it drives the wheel as a rack would drive a pinion, the only difference being that the rack would have a motion of translation while the worm, being held so that it may have no endwise motion, is forced to rotate.

In plane A–A the teeth are involute. In other parallel planes, conjugate action necessitates varying the form of the tooth.

The speed ratio of worm and wheel is obtained in exactly the same fashion as in the case of screw and nut.

When the worm makes one complete revolution—assuming that it is single-threaded—the wheel is advanced one tooth distance.

The speed ratio

$$\frac{\omega_F}{\omega_D} = \frac{1}{N}$$

where

ω_F is the angular velocity of the wheel
ω_D " " " " " " worm
N is the number of teeth on the wheel.

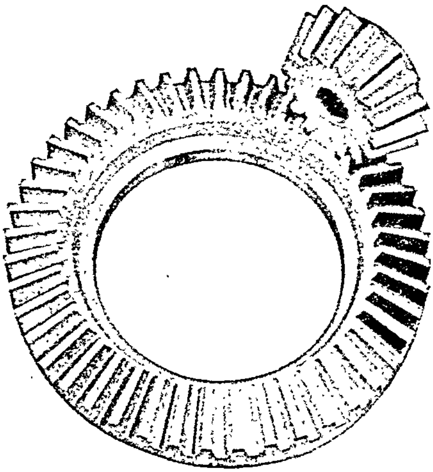
If the worm is double-threaded the speed ratio becomes $\frac{2}{N}$, etc.

81. Bevel Gearing. In rolling contact the drive between intersecting shafts was supplied by the use of rolling cones. Just as the rolling cylinders for connecting parallel shafts were developed into toothed wheels, or gears, the rolling cones may be used as the pitch surface basis and developed, by supplying teeth on these surfaces, into *bevel gears* when the shafts are intersecting.

And just as the evolution of the rolling cylinders demanded that the gears produce the same action as the pure rolling contact of the cylinders, the modification of the cones must result in pure rolling contact of the cones along their elements.

The elements of velocity relationship of the rolling cones are not disturbed by their development into gears, and the conclusions derived concerning these elements in Art. 56 are valid.

The terms of a gearing vocabulary in the case of bevel gears include those fundamental definitions which were discussed in Art. 65 as well as some additional ones which are illustrated in Fig. 212.



Courtesy The Gleason Works

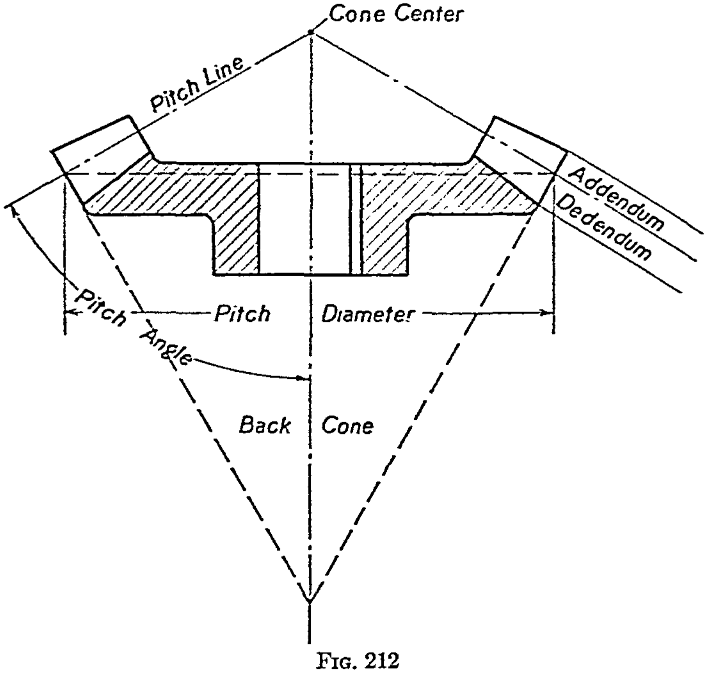


FIG. 212

The theory of bevel gearing borrows for the most part from the analogous situations in spur gearing.

There is one distinction, however, which must be made.

A bevel gear has a plane motion of pure rotation about its own axis. The

relative motion of a pair of bevel gears is, however, spherical. Let us examine this statement, aiding our observations with Fig. 213.

The cone Abc represents the base cone of a bevel gear, that is, the cone whose base is at the level of the base circle of an involute spur gear of equivalent pitch circle and pressure angle. The location of this base cone is discussed in the illustrative example which follows. We may imagine this cone to be covered with a thin skin. If we cut the skin along any line Ad, and start unwrapping or peeling the skin, keeping it always taut and allowing

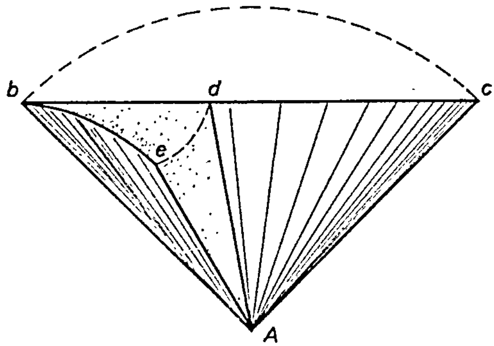


Fig. 213

point d to come forward to any position like e, the surface Ade which is generated by line Ad as it sweeps out will be the surface forming the tooth. This is very similar to the process we employed in generating the involute of our spur gears. In that case we unwrapped a taut cord from a base circle. Now we are unwrapping a taut covering from a base cone. The point d is remaining ever at the same distance from A, and therefore moves on the surface of a sphere. The curve de is called a *spherical involute.*

The use of a sphere as basis makes it impossible to show, in a drawing which must be made in one plane, a development of the involute tooth surface.

It is customary, therefore, to use an approximation, which is known as *Tredgold's* approximation.

The cone Apm (Fig. 214) is the pitch cone of the gear. The cone Bpm is its back cone, erected with all of its elements, like Bp, perpendicular to corresponding elements, like Ap, of the pitch cone.

The spherical surface whose outline is shown as mSp is tangent to the back cone at circle mp, which is the common base of the pitch and back cones. Tredgold's approximation substitutes this back cone surface for the spherical surface. Within the limits of surface actually used for teeth, a comparatively short distance, the approximation is very close, and the entire tooth profile practically coincides with the surface of the sphere.

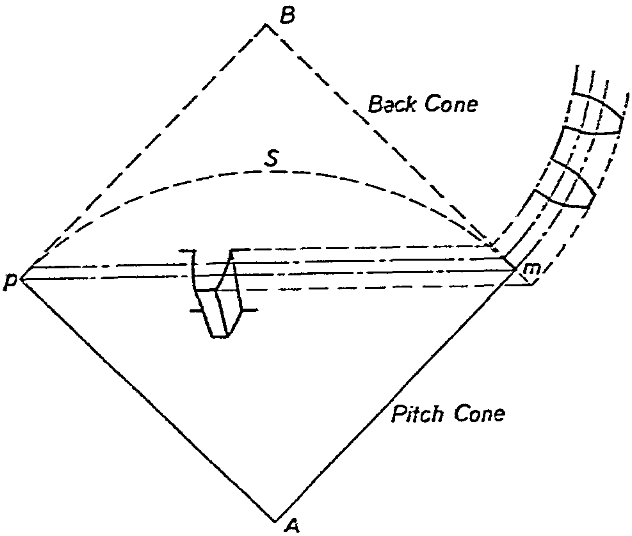


Fig. 214

In the application to the layout of bevel-gear teeth, we lay out the teeth on the developed surface of the back cone. Such a surface can be drawn since this development becomes a plane surface. Then, by projection in the drawing, we wrap the development back to its original location on the back cone. This will fix the end profiles of the teeth and converging straight lines drawn to A will generate the tooth surfaces.

When the outline of the base of the back cone (circle mn) has been developed, we have a pitch circle and the teeth are laid out upon it in exactly the same fashion as for any involute spur gear.

While the use of standard involute dimensions has some advantages, bevel gears are designed in mated pairs, and dimensions whose purpose is to provide interchangeability have little advantage. Improved performance in the form of quieter action and stronger teeth has been secured by departing from these standards and developing a system, known as the *Gleason* system, which has been adopted by the American Gear Manufacturers Association. In it pressure angles of $14\frac{1}{2}°$, $17\frac{1}{2}°$, and $20°$ are used and a series of addendum values based upon the differing speed ratios to be employed is used.

Bevel gears must be designed in matched pairs. As an illustrative example, we shall follow through the steps in the design of a pair, having standard involute teeth of $14\frac{1}{2}°$ obliquity, and diametral pitch = 2. The gear has 30 teeth, and the pinion 24. The shafts are 90° apart (Fig. 215). Axes Oa and Ob are drawn at 90°. On Oa, the pitch radius of the gear ($Oc = \frac{30}{2\,\text{D.P.}}$

$= \frac{30}{4} = 7.5$ in.) and on Ob the pitch radius of the pinion ($Od = \frac{24}{2\text{ D.P.}} = \frac{24}{4}$ $= 6$ in.) are laid off. If ce and de are drawn parallel to the axes they intersect at point e, and the pitch cones of the gears have been established. It will be noted that these pitch cones have an angular velocity equal to that obtained from the numbers of teeth.

At right angles to Oc we draw line feg with eg, which is the addendum distance $= 0.50$ in., and ef, the dedendum distance $= 0.58$ in. The section

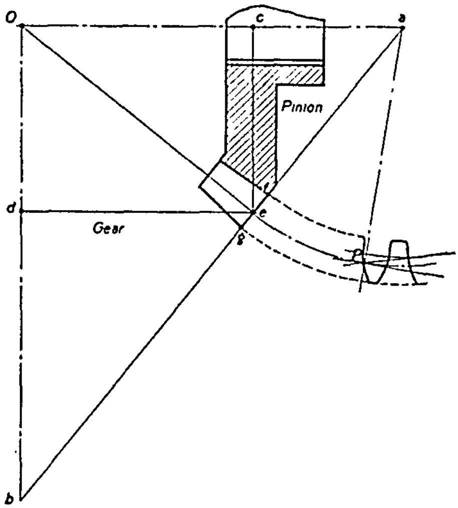


FIG. 215

is now completed. If the teeth are to be drawn, Tredgold's approximation is used. The construction is shown. Line feg, produced to meet Oa and Ob, establishes the back or normal cones.

With center at a, and radius ae, the pitch circle of the pinion at the level e is drawn. If any point, like P, is selected to serve as pitch point, the base circle may be located and involute teeth drawn in equivalent manner to that employed for any involute spur gear.

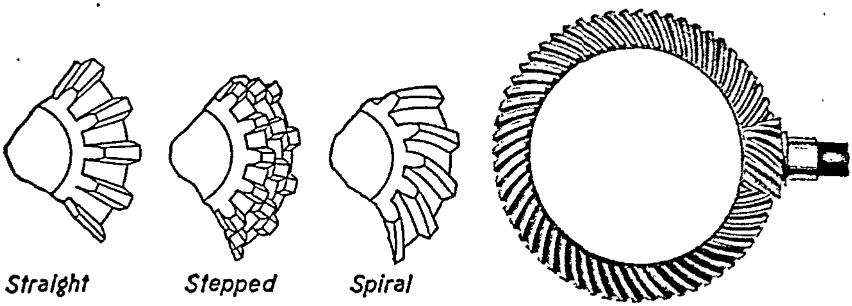
These developments may be used, as in spur gears, to check for interference, paths of contact, etc.

The angle between the axes of bevel gears is most frequently found as 90° but may be greater or smaller.

Bevel gears having an angle of more or less than 90° are known as *angular bevels.* When the angle is 90° and the two gears are equal, the gears are called *miter gears.*

A bevel gear in which the pitch cone has become a plane is known as a *crown gear.*

82. Spiral Bevel Gears. In the bevel gears just discussed the elements of the tooth surfaces are straight lines converging at the apex of the cone. A smoother and quieter action is produced by departing from the straight-



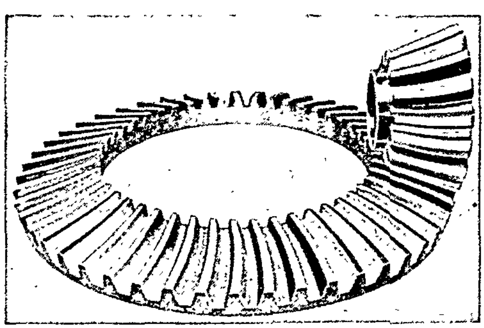
Courtesy The Gleason Works

FIG. 216

line teeth to the *spiral* type shown in Fig. 216. This development is similar to the progress from spur gear through stepped wheels to helical forms. In this case, concentric "steps" of straight bevel gears are advanced beyond each other to obtain spiral bevels.

The action is improved, as it was in the case of the substitution of helical gears for spur gears with straight-line teeth, in that we again have gradual engagement, contact beginning at one end of the gear tooth and gradually progressing across the face.

The elements of theory which were applied to straight-tooth bevels apply again in the case of spiral bevel gears. A recent modification is the



Courtesy The Gleason Works

FIG. 216-a

"Zerol" bevel gear, which has zero-degree spiral angle. These gears combine the localized tooth contact of spiral bevels with the favorable thrust load characteristics of straight bevel gears. They are illustrated in Fig. 216-a.

83. Hypoid Gears are used to connect non-intersecting shafts. They have become of interest because of their application in automotive design. These gears are a modification of hyperboloidal gears. Hyperboloids of revolution are shown in Fig. 217.

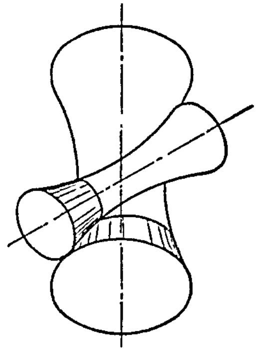
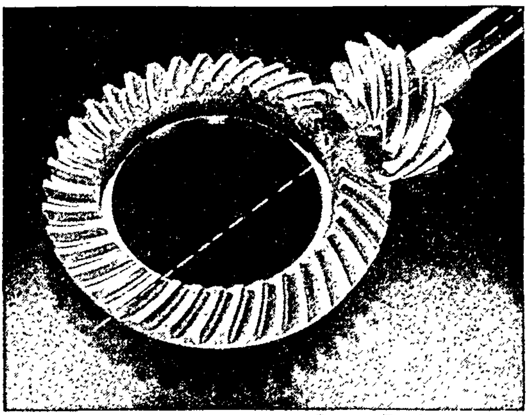


FIG. 217

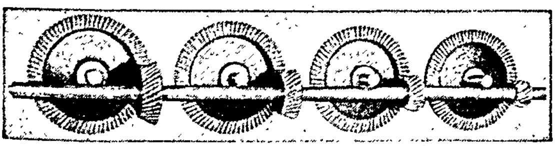
The shaded frustums of the figures indicate the portions which would form the pitch surfaces of hyperboloidal gears. These frustums bear a strong resemblance to cones, and in hypoid gears, it is the cones which form the approximations which are generally used.

Figure 218-a shows a pair of hypoid gears. It will be noted that the pinion axis is offset from the gear axis, a factor of importance which has been used in lowering the bodies of automobiles. The offsetting of the pinion axis from the gear axis makes it possible for the shaft of the pinion to be continuous as it passes the gear axis, and this feature is of



Courtesy The Gleason Works

FIG. 218-a



Courtesy The Gleason Works

FIG. 218-b

no mean importance in arranging multiple drives from one shaft, as in Fig. 218-b. In addition, a larger and stronger pinion becomes available than is the case with spiral bevel gears, which is the type of gearing bearing the strongest resemblance to hypoids.

PROBLEMS

250. Design a pair of involute bevel gears, diametral pitch = $1\frac{1}{2}$, 26 teeth and 40 teeth, shafts at right angles. Pressure angle = $14\frac{1}{2}°$.

251. Design a pair of involute bevel gears, diametral pitch = 3, 24 teeth and 48 teeth, pressure angle = $14\frac{1}{2}°$. Shafts at right angles.

252. A pair of helical gears connects two shafts 45° apart. Speed ratio = 2.5 : 1. Center distance = 5 in. Assuming any proper combination of helix angles, determine the corresponding pitch diameters. If 50 and 20 teeth are used, determine the normal diametral pitch of the gears.

253. A pair of helical gears connects two shafts 90° apart. Speed ratio = 2 : 1. One gear has a pitch diameter of 8 in. and a helix angle of 30°. Find the helix angle and pitch diameter of the second gear.

254. A worm gear is double-threaded and meshes with a wheel of 44 teeth. If the wheel is to turn at 10 r.p.m., what must be the speed of the worm?

84. Gear Trains. A train of gears is a series of connected pairs of gears. The over-all or total speed ratio made available usually exceeds the limits imposed by practical considerations when a single pair of gears is used. The theoretical considerations which we have discussed for contacting driver and follower rarely fix limits of speed ratio, but we should find ourselves in difficulty when, in designing a pair of spur gears, we attempt such speed ratios as 200 : 1. The difficulty would be a practical one in that the size of one gear would then be so great that its construction might be expensive and difficult, and provision for its location within the confines of the intended machine impossible. These very practical considerations will limit the size, in spite of the fact that the theory of gearing might permit the design.

We turn, then, to combinations, or series of gears, which will yield the desired speed ratio, while at the same time demanding only such sizes of individual gears as the usual practical considerations will permit.

In addition to the possibilities of speed ratio, gear trains provide flexibility in the drive, which is of tremendous value in the design of machines. For example, as in the case of the automobile transmission, we may design a *selective* drive—that is, one in which selection of different speed ratios may be readily made. In this case, and in many machine drives, *reversibility* of the relative direction between driver and follower is a necessary feature, and, again, provision of a gear train will provide the desired choice of direction.

In other cases, *multiplicity* of the drive is essential. A single driver frequently serves as the source of drive for several follower shafts, and a gear train may be used to transmit the desired magnitude and direction of velocity from the common driver to the individual followers.

In Art. 59 the term *train value* has been established as the ratio of the angular velocity of the last wheel, or follower, of a series to the angular

velocity of the first wheel, or driver. In that case the element of mechanism was the rolling cylinder, and the train value was established as a ratio of diameters.

In gear trains speed ratios are established as the ratio of numbers of teeth.

Figure 219 furnishes a basis from which we may derive our concept of train value in gear trains. This series of gears is similar to the series of rolling

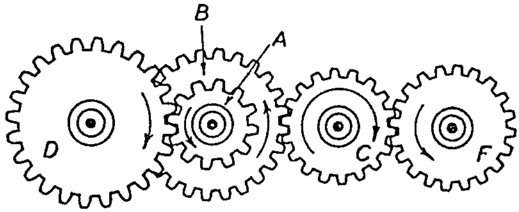


FIG. 219

cylinders which was discussed in Art. 59. This is a proper approach, for we know that the addition of teeth to rolling cylinders insures positive drive, but retains the velocity features of the rolling cylinders which have now become the pitch cylinders of the gears.

The speed ratio of the train, that is, the ratio of the angular velocity of the final gear, or follower, to the angular velocity of the initial gear, or driver, will again be called the train value, and conventionally symbolized as t.v.

Then, by definition,

$$\text{t.v.} = \frac{\omega_F}{\omega_D}.$$

In the first pair of gears, D and A, D is the driver and A is a follower. Then

$$\frac{\omega_A}{\omega_D} = \frac{T_D}{T_A}$$

where T_D is the number of teeth on the driver
and T_A " " " " " " " follower.

But gears A and B are keyed to the same axis, and therefore must have the same angular velocity, or $\omega_B = \omega_A$.

Proceeding to the next pair of gears B and C, we find B acting as driver and C as follower.

Then $$\frac{\omega_C}{\omega_B} = \frac{T_B}{T_C}.$$

The next pair consists of gear C, now acting as driver and gear F as follower.

Then
$$\frac{\omega_F}{\omega_C} = \frac{T_C}{T_F}.$$

If we now gather the terms of the train value, by multiplying the corresponding sides of the equations, we find

$$\frac{\omega_A}{\omega_D} \times \frac{\omega_C}{\omega_B = \omega_A} \times \frac{\omega_F}{\omega_C} = \frac{\omega_F}{\omega_D} = \text{t.v.}$$

and
$$\text{t.v.} = \frac{T_D}{T_A} \times \frac{T_B}{T_C} \times \frac{T_C}{T_F}.$$

The term T_C will cancel, since a gear acting as an idler presents its number of teeth as a term in both the numerator and denominator of the train value. The purpose of such an idler is to reverse the direction of the follower.

We may summarize the train value of a gear train by the following statement:

The train value of a gear train is equal to the product of the numbers of teeth on all drivers divided by the product of the numbers of teeth on all followers.

The directional relationship of the train value must be established, as in the case of rolling cylinders, by following through the series, as has been done in Fig. 219 with arrow-heads. We discover in this manner that the follower and driver of this train are turning in opposite directions.

When follower and driver have the same direction, or sense, of rotation, we shall consider the train value to be positive (+); when they have opposite sense, the train value is negative (−). Such a convention will simplify the giving of data in problems involving gear trains.

PROBLEMS

255. Determine the train value of the textile machine drive shown. Number of teeth on each gear is given. The 11- and 44-tooth gears are fastened together, or "compounded," and the 16- and 40-tooth gears are compounded. *Ans.* −21.3

256. If the driver, D, of Problem 255 has an angular velocity of 1600 r.p.m., clockwise, determine

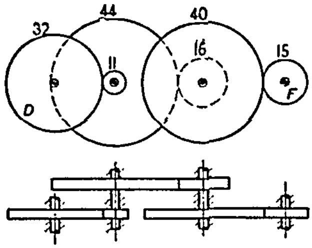
(a) The angular velocity of the 11-tooth gear.

(b) The angular velocity of the 16-tooth gear.

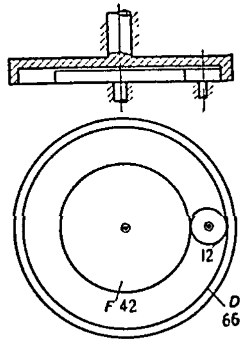
(c) The angular velocity of the follower, F.

257. A machine tool drive contains the speed-changer shown. The numbers of teeth are given. Determine the train value.

258. If the drive of Problem 257 is reversed so that the 42-tooth gear rotating at 360 r.p.m., counter-clockwise, becomes the driver, determine the angular velocity of the 66-tooth annular.



Prob. 255



Prob. 257

259. In an automatic textile-printing machine three follower shafts F_1, F_2, and F_3 are driven by the driver D which rotates at 2400 r.p.m.

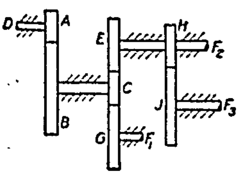
Determine the angular velocity of each follower shaft. Gear A has 20-teeth, B—60, C—22, E—36, G—42, H—28, J—52. *Ans.* $F_1 = +419$; $F_2 = +489$; $F_3 = -263$

260. If the diametral pitch of gears A and B (Problem 259) = 2, that of E, C, and G = 3, and that of H and J = $1\frac{1}{2}$, find the layout distances for all shaft axes relative to the axis of D.

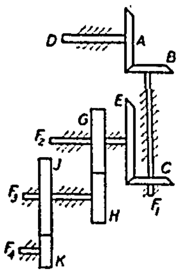
261. The drive shaft, D, rotates at 200 r.p.m. Find

(a) The "over-all" train value (D to F_4)
(b) The speed of F_1
(c) " " " F_2
(d) " " " F_3
(e) " " " F_4

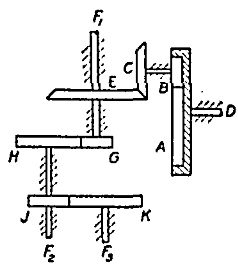
Gear A has 42 teeth, B—30, C—32, E—56, G—42, H—34, J—50, K—22.



Prob. 259



Prob. 261



Prob. 262

262. Three machines are to be driven from a motor keyed to shaft D. Motor speed = 720 r.p.m. Find

(a) The train value—D to F_3
(b) The speed of F_1
(c) " " " F_2
(d) " " " F_3

Gear A has 72 teeth, B—20, C—28, E—50, G—20, H—44, J—26, K—48.

4 pairs: $\quad$ t.v. $= \dfrac{96}{12} \times \dfrac{96}{12} \times \dfrac{96}{12} \times \dfrac{96}{12} = 4096 : 1 =$ Too high.

The four pairs, yielding a train value of 4096 : 1, must now be modified to reduce the train value to 600 : 1.

The simplest modification will consist of leaving three pairs intact, and determining the necessary ratio of the fourth pair.

If we then set up, as follows,

$$8 \times 8 \times 8$$

we should add a pair of ratio 600 : 512, thus

$$8 \times 8 \times 8 \times \frac{600}{512} = 600 : 1$$

No gears of 600 or of 512 teeth, respectively, are available under the imposed maximum limit of 96 teeth.

Reducing the fraction

$$\frac{600}{512} = \frac{75}{64}$$

will reduce the required tooth numbers to within the limit.

Our final selection is the following train:

$$\frac{96}{12} \times \frac{96}{12} \times \frac{96}{12} \times \frac{75}{64}$$

which has a train value of 600 : 1.

Other combinations are possible. We have modified but one possible pair; all of the pairs might be modified, or any combination employed. In general, it is advisable to have a uniform speed ratio prevail throughout the series.

A combination like the following

$$\frac{60}{12} \times \frac{60}{12} \times \frac{60}{12} \times \frac{72}{15}$$

is a more favorable one in that a nearly uniform speed ratio is maintained throughout the series. The basis of trial here is the selection of pairs which most nearly approximate the fourth root of the train value, since four pairs are to be used and the train value is the product of the four speed ratios.

The train selected has given a satisfactory distribution of the train value over a series of speed ratios. No effort has been made as yet to fix the proper directional relationship of follower to driver.

Then, to satisfy the condition that the shafts must remain parallel, we note that the distance between the centers of shafts S and S_1 (Fig. 224) is

$$c = R_J + R_K$$

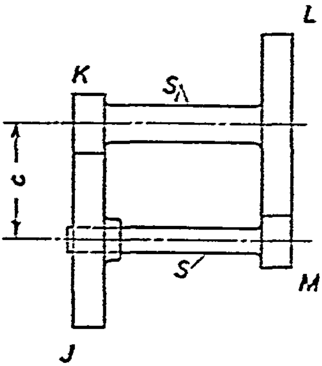


Fig. 224

where R_J is the pitch radius of gear J
and R_K is the pitch radius of gear K

$$R_J = \frac{1}{2}\frac{T_J}{\text{D.P.}}, \qquad R_K = \frac{1}{2}\frac{T_K}{\text{D.P.}}$$

Then
$$R_J + R_K = \frac{T_J + T_K}{2\ \text{D.P.}} = c.$$

In like manner

$$R_L + R_M = \frac{T_L + T_M}{2\ \text{D.P.}} = c.$$

Then
$$\frac{T_J + T_K}{2\ \text{D.P.}} = \frac{T_L + T_M}{2\ \text{D.P.}}$$

or
$$T_J + T_K = T_L + T_M.$$

This sum of $T_J + T_K$, or $T_L + T_M$, is now taken as a number called the *multiple number* N.

Now, for our other condition, the train value.

The given train value, 20 : 1, is first divided into two numbers, preferably as nearly alike as possible. The ideal condition would be $\sqrt{20}$. These two numbers will be the speed ratios of the two pairs of gears, so that a rational number which in turn will produce integral numbers of teeth is more practical than $\sqrt{20}$.

We therefore select the numbers 4 and 5, whose product is the train value 20.

The speed ratio 4 : 1 is assigned to the first pair:

$$4 = \frac{T_J}{T_K}$$

and
$$5 = \frac{T_L}{T_M}.$$

Now the multiple, N, may be established,

since
$$N = T_J + T_K = T_L + T_M$$

N must be a sum which can be split into two parts in the ratio of $\frac{4}{1}$ and also into two parts in the ratio $\frac{5}{1}$.

Such a number is 30, the least common multiple of $4 + 1$ and $5 + 1$.

With multiple of 30, we substitute in our simultaneous equations:

$$30 = T_J + T_K$$

and

$$4 = \frac{T_J}{T_K}$$

Then $T_J = 24$ and $T_K = 6$

Also

$$30 = T_L + T_M$$

$$5 = \frac{T_L}{T_M}$$

Then $T_L = 25$ and $T_M = 5.$

This selection of four gears would produce the desired train value, and also keep the shafts parallel. We have, however, encountered an obstacle—both gears K and M have numbers of teeth below the announced minimum.

We therefore select as N a higher common multiple than the least common one, 30.

Trying 60 as the next stage,

$$60 = T_J + T_K$$

$$4 = \frac{T_J}{T_K}$$

$$T_J = 48 \quad \text{and} \quad T_K = 12$$

$$60 = T_L + T_M$$

$$5 = \frac{T_L}{T_M}$$

$$T_L = 50 \quad \text{and} \quad T_M = 10.$$

Again we have failed to satisfy the limitation of minimum number of teeth = 12.

We try the next higher common multiple of $N = 90$.

Now

$$90 = T_J + T_K$$

$$4 = \frac{T_J}{T_K}$$

$$T_J = 72 \quad \text{and} \quad T_K = 18.$$

Also

$$90 = T_L + T_M$$

$$5 = \frac{T_L}{T_M}$$

$$T_L = 75 \quad \text{and} \quad T_M = 15.$$

All of the gears now have numbers of teeth which exceed the minimum limit.

In the interests of good design procedure we should check the selection of tooth numbers.

The train value will be

$$\frac{72}{18} \times \frac{75}{15}$$

which does equal 20 and we have proper speed ratio.

The distance, c, between the shafts should also be checked to insure their remaining parallel.

$$c = \frac{T_J + T_K}{2\ \text{D.P.}} \quad \text{or} \quad \frac{T_L + T_M}{2\ \text{D.P.}}$$

Then
$$c = \frac{72 + 18}{2\ \text{D.P.}} \quad \text{or} \quad \frac{75 + 15}{2\ \text{D.P.}}.$$

Since all gears have the same diametral pitch, and

$$\frac{72 + 18}{2} = \frac{75 + 15}{2}$$

the shafts are parallel.

When the two pairs of a reverted gear train are of *different* diametral pitch the method must be modified.

Illustrative Example. Let us again design a reverted gear train for a train value of 20 : 1, with minimum number of teeth = 12. One pair of gears is to have diametral pitch = 2, the other pair has diametral pitch = 3.

If, proceeding as before, we establish a common multiple of 30, we may modify our previous method as follows:

$$c = \frac{T_J + T_K}{2\ \text{D.P.}_{\cdot 1}} = \frac{T_L + T_M}{2\ \text{D.P.}_{\cdot 2}}$$

Then
$$\text{D.P.}_{\cdot 2}\ (T_J + T_K) = \text{D.P.}_{\cdot 1}\ (T_L + T_M).$$

The previous least common multiple, $N = 30$, was a sum of $T_J + T_K$ or of $T_L + T_M$. Multiplying it by an integer, which we note is the diametral pitch, will not affect its ability to be divided into numbers in the ratio of 4 : 1 or 5 : 1.

We therefore set

$$\text{D.P.}_{\cdot 2}\ (T_J + T_K) = \text{D.P.}_{\cdot 1}\ (T_L + T_M) = 30$$

or any higher multiple of 30.

In this case let us try 180, a higher common multiple.

$$\text{D.P.}_2\,(T_J + T_K) = 180$$

$$3\,(T_J + T_K) = 180$$

$$\frac{T_J}{T_K} = 4$$

Then $\qquad T_J = 48 \qquad$ and $\qquad T_K = 12.$

$$\text{D.P.}_1\,(T_L + T_M) = 180$$

$$2\,(T_L + T_M) = 180$$

$$\frac{T_L}{T_M} = \frac{5}{1}$$

Then $\qquad T_L = 75 \qquad$ and $\qquad T_M = 15.$

Let us next check this selection.

The train value

$$\frac{48}{12} \times \frac{75}{15} = 20.$$

The distance between axes of gears J and K

$$c = \frac{T_J + T_K}{2\,\text{D.P.}_1} = \frac{48 + 12}{2 \times 2} = 15 \text{ in.}$$

The distance between axes of gears L and M

$$c = \frac{T_L + T_M}{2\,\text{D.P.}_2} = \frac{75 + 15}{2 \times 3} = 15 \text{ in.}$$

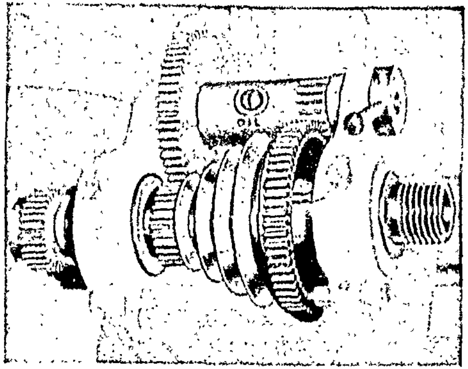
The shafts are, therefore, parallel.

Selective gear trains are those in which the drive may be shifted so that different speed ratios are available.

The reverted gear train which we have just analyzed is a simple form of selective train. This is shown in greater detail in Fig. 225, an example taken from an engine lathe.

Shaft S is continuous through gears J and M. J is keyed to pulley P but floats freely on shaft S. A pin is provided which may be put in place to lock gear M to pulley P. Now the driving pulley P is directly connected with shaft S, and they will rotate at the same speed.

Gear M is always keyed to shaft S, and gears K and L are keyed upon a parallel shaft in eccentric bearings so that this shaft may be placed in the position shown in the figure, where K is in mesh with J, and L with M; or this parallel shaft may be dropped back, so that K and L are no longer in contact with their mates.



Courtesy The Atlas Press Co.

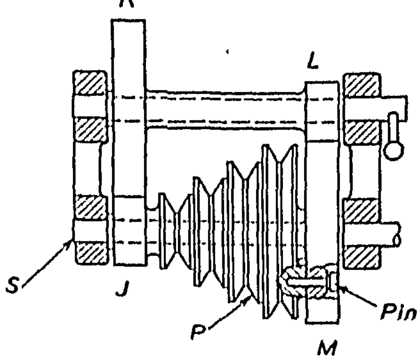
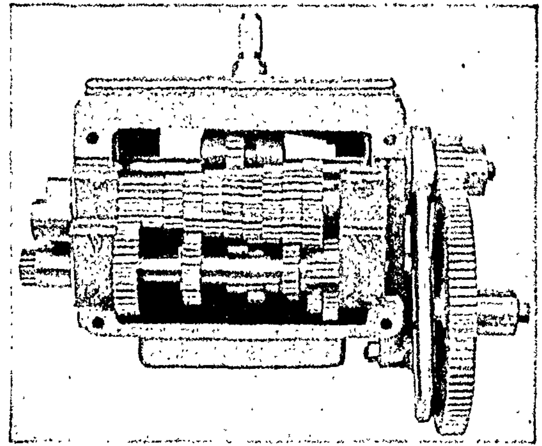


FIG. 225



Courtesy The R. K. LeBlond Mach. Tool Co.

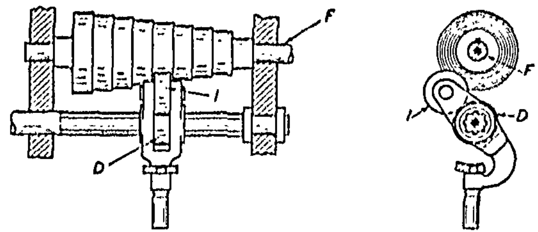


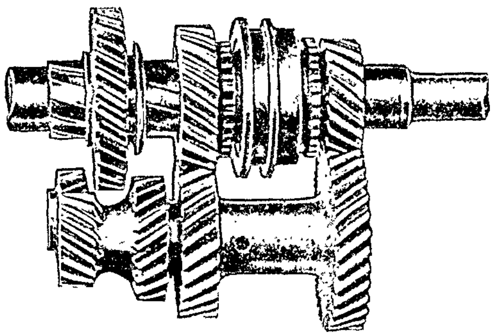
FIG. 226

If we unlock the pin joining M with P, and bring the shaft carrying K and L up so that contact is made between J and K, and L and M, the drive, originating at the pulley P, goes through gear J to K and L, and through L to M, which rotates at the speed of the shaft S.

This gear train affords choice of two speeds for shaft S, which is the spindle of an engine lathe. Since the pulley itself has four steps for a belt drive, a total opportunity of selecting one of eight different speeds is available from a constant speed belt.

Another form of selective train is shown in Fig. 226. The gear D acting as driver may be moved to any desired position along its shaft, and the drive sent through the idler I to any one of the gears in the series mounted upon shaft F. We now have an opportunity of selecting any one of eight available speeds.

The automobile transmission illustrated in Fig. 227 is another form of selective drive. In addition to a direct connection between gear a and



Courtesy Chrysler Corporation

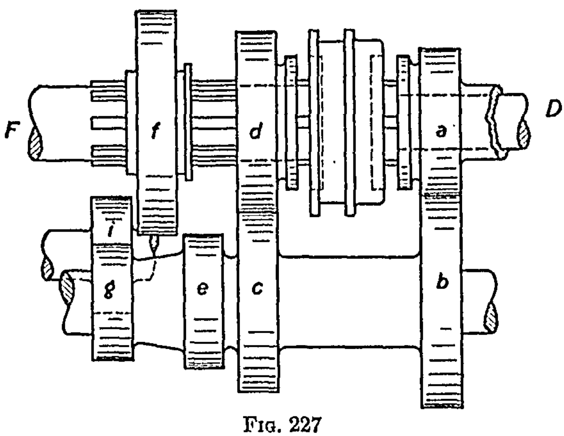


Fig. 227

shaft F, we may select a drive through a to b and c to d (which will drive F) which is one forward speed. Or we may elect to have the drive go from a to b and e to f (driving F), an additional forward speed.

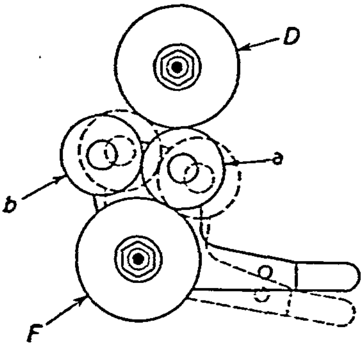


FIG. 228

If the drive is sent from a to b and g to i (an idler) and then on to f we have one reverse speed.

Gear trains to provide *reversal* of direction may be one of several types. One reverse train is shown in Fig. 228. By shifting the handle the drive may be established through one of the two possible paths D to a to F or D to b to a to F. D and F are mounted on fixed axes.

When the path of drive selected is D to a to F, a acts as an idler, and the driver and follower have the same sense of rotation.

When the drive is transmitted from D to b to a to F, two idlers, a and b, have been placed in the train, and driver and follower have opposite senses of rotation.

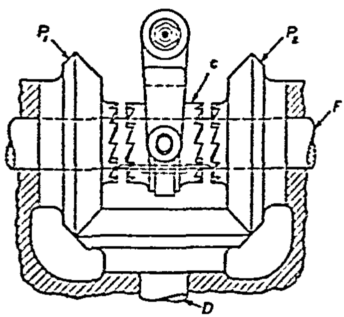


FIG. 229

The *bevel-gear reversing mechanism* is illustrated in Fig. 229.

The toothed clutch c is free to slide along the axis of shaft F but is keyed to it, so that both must rotate together.

The driver, D, is in contact with two pinions, P_1 and P_2, which float freely upon the shaft, and which have opposite senses of rotation. If the clutch handle is moved so that the teeth of the clutch engage the teeth on the clutch end of pinion P_1, the follower shaft F is driven by pinion P_1. By moving the clutch handle in the opposite direction, the drive is transmitted from D through pinion P_2 to the follower, reversing the sense of shaft F. In the mid-position of the clutch handle, a neutral position is established, and the follower shaft remains at rest.

With the exception of the bevel-gear reversing train, all of the gear trains which have been discussed have served to connect driver and follower on parallel shafts.

Non-parallel shafts may also be connected by gear trains. The train value analysis follows the same procedures which we have been employing, and the numbers of teeth fix the several speed ratios. Directional analysis may

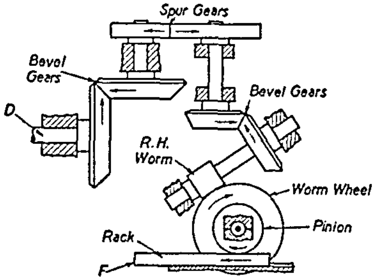


FIG. 230

best be made by following through on a sketch, noting the direction of the linear velocity of each pair of gears at their pitch point. One example of such a drive is shown in Fig. 230.

PROBLEMS

272. Design a back gear train. The train value is to be 18 : 1. All gears are of the same diametral pitch. No gear is to have fewer than 12 teeth, or more than 90 teeth.

Ans. $T_A = 20$; $T_B = 90$; $T_C = 22$; $T_E = 88$.

273. Solve Problem 272, with train value changed to 24 : 1.

274. Solve Problem 272, with train value changed to 16 : 1.

275. Solve Problem 272, with train value changed to 17 : 2. In this case, gears A and B are to have diametral pitch = 3 and gears C and E are to have diametral pitch = 4. The minimum number of teeth on any gear is to be 12.

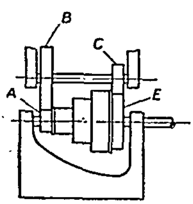
276. The drive shaft, D, has a speed of 220 r.p.m. Gears A, C, and G are keyed to the drive shaft. Gears B, E, and H are supported on a shaft parallel to D, but rotate freely on the shaft.

The speeds of B, E, and H are to be 55, 380, and 1980 r.p.m., respectively. Design the train, selecting gears from the stock sizes given in Problem 268. All gears are of the same diametral pitch.

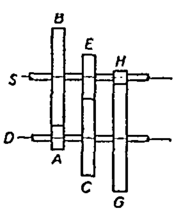
277. Solve Problem 276, with data changed as follows: diametral pitch of A and B = 3, diametral pitch of C and E = 2, and diametral pitch of G and H = 4. What is the distance between centers of the parallel shafts?

278. The pointers s, m, and h represent the second, minute, and hour hands of a clock mechanism. Design the train of gears.

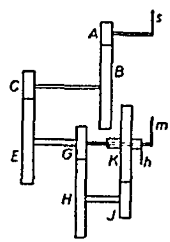
The minute hand, m, has the angular velocity of gear G, the hour hand, h, the angular velocity of gear K, and the second hand, s, the angular velocity of gear A.



PROB. 272



PROB. 276



PROB. 278

87. Epicyclic Gear Trains. All of the trains of gears which have been discussed contained only individual gears whose axes remained fixed in location.

In modern machine design, we encounter an increasing use of a gear train which contains individual gears, some of which have axes which are themselves in motion.

Let us very carefully study the motion which such gear trains involve, in order that we may establish true understanding, rather than take refuge in formulas or stereotyped procedure which might here become a subterfuge. If the conclusions we reach are capable of being codified into expression as formulas for application to definite problems, we shall so codify them, but not before we have built a firm foundation. Engineering methods of attack are frequently summarized in the form of expressions like formulas, which are convenient tools for application in problems. Unless, however, the user of formulas has an appreciation of their background or derivation, he may apply formulas falsely. A sound philosophy of engineering education will embrace an active and constant search for truth and understanding, and will not permit the type of evasion represented by the unintelligent "formula substituter."

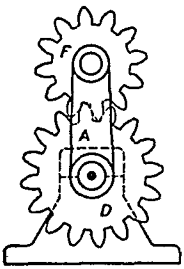


FIG. 231

An *epicyclic gear train* is illustrated in Fig. 231. The gear D is mounted upon the same fixed axis as arm A, but is not fastened to it, so that D and A may turn freely relative to each other.

Gear F is mounted upon a pin carried by the arm. When the arm is rotated about its fixed axis, gear F will be carried around the circumference of D. At the same time, gear F is free to rotate upon its own axis. This form of motion is such as we observe in astronomical bodies, where a planet may rotate about its own axis, while that axis itself moves in some orbit relative to another body. For this reason, such gear trains are frequently called *planetary trains.*

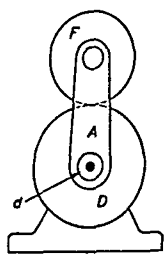


FIG. 232

Gears D and F are represented by the rolling cylinders shown in Fig. 232. These have diameters which are equivalent to the pitch diameters of D and F.

Here we find a mechanism in which our previous examinations of absolute and relative motion (Art. 49) furnish the surest footing as a basis of study.

We noted in Art. 49 that the *relative angular velocity* of two rotating bodies is the *difference* between their *absolute angular velocities.*

Wheel D of Fig. 232 and the arm A are two bodies rotating on the same

fixed axis, d. The two are not joined and their absolute angular velocities may be the same or different.

If we call the absolute angular velocity of D, ω_D, and the absolute angular velocity of A, ω_A, then the angular velocity of D relative to the arm is the difference between ω_D and ω_A. This is a vector difference. Let us call this velocity of D relative to A, $\omega_{D/A}$.

Then $$\omega_{D/A} = \omega_D \rightarrow \omega_A.$$

Now let us consider the relationship between F and A.

Here the angular velocity of F relative to the arm,

$$\omega_{F/A} = \omega_F \rightarrow \omega_A.$$

Then $$\frac{\omega_{F/A}}{\omega_{D/A}} = \frac{\omega_F \rightarrow \omega_A}{\omega_D \rightarrow \omega_A}$$

or $$\frac{\text{Angular velocity of } F \text{ relative to } A}{\text{Angular velocity of } D \text{ relative to } A}$$

$$= \frac{\text{Absolute angular velocity of } F - \text{Absolute angular velocity of } A}{\text{Absolute angular velocity of } D - \text{Absolute angular velocity of } A}.$$

$$\frac{\omega_{F/A}}{\omega_{D/A}} = \frac{\text{Angular velocity of } F \text{ relative to } A}{\text{Angular velocity of } D \text{ relative to } A}$$

is an expression of the relative velocities of the two gears to each other, which is fixed by the numbers of teeth on the gears.*

If a 48-tooth driver meshing with a 32-tooth follower is mounted upon an arm which is itself at rest as in Fig. 233, their speed ratio relative to each other is $-3:2$.

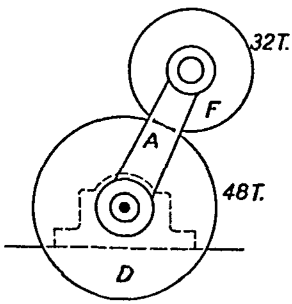


FIG. 233

If we now imagine that the arm is itself in motion, this ratio of relative speeds, fixed only by the tooth numbers, is still $-3:2$. The absolute motion of the gears is affected by giving the arm an absolute motion, but the ratio of their speeds relative to each other remains unchanged.

* When two bodies, A and B, are in motion their relative velocity may be measured by refering them to a third body for convenience. The third body may itself be in motion or at rest. In either event, the relative velocity between A and B remains unchanged. For example, in measuring the relative velocity between two bodies, we customarily take the difference between their absolute velocities. This procedure makes use of a third body—the earth, which is assumed to be fixed. Actually the earth is in motion, but its motion does not affect the relative velocity between the other two bodies. Using the arm as a third body for convenient reference does not affect the relative velocity of the two gears, which is dependent only upon their numbers of teeth.

If, as in the past, we call this ratio of relative speeds the train value, then

$$\text{t.v.} = \frac{\text{Number of teeth on } D}{\text{Number of teeth on } F}$$

or, in the case of a multiple epicyclic train like that of Fig. 234, the train value

$$\text{t.v.} = \frac{\text{Product of teeth on all drivers}}{\text{Product of teeth on all followers}}.$$

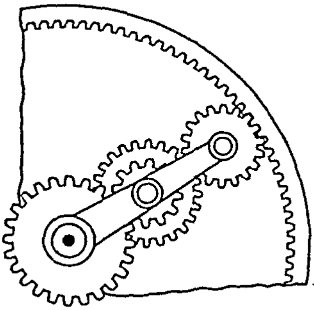


FIG. 234

Now, substituting this term in our original statement of the relationship of the absolute angular velocities of the epicyclic gear train, we have

$$\text{t.v.} = \frac{\omega_F - \omega_A}{\omega_D - \omega_A}.$$

This expression is the tool with which we analyze the motion of epicyclic gear trains. Its derivation has rested firmly on valid principles of absolute and relative motion, and its final form may be used with confidence, if its derivation has been mastered.

We shall illustrate its use by turning to the gear train of Fig. 233 which has the following description.

Gear D has 48 teeth, and an absolute angular velocity of 100 r.p.m., clockwise.

Gear F has 32 teeth.

Arm A has an absolute angular velocity of 50 r.p.m., clockwise.

The absolute angular velocity of gear F is to be determined.

We first note the presence of the arm. Then gear F has an axis which is itself rotating, and the gear train is an epicyclic one, to which our expression for train value does apply.

$$t.v. = \frac{\omega_F - \omega_A}{\omega_D - \omega_A}.$$

The train value is the speed ratio of the two gears relative to the arm, which we can determine by noting the speed ratio which the two gears would have if the arm were held still.

This would be

$$t.v. = -\tfrac{48}{32} = -\tfrac{3}{2}.$$

The minus sign of the train value indicates that the two gears will have opposite senses of angular velocity if the arm is held still.

Then, if we call clockwise absolute angular velocity positive (+)

$$t.v. = -\frac{3}{2} = \frac{\omega_F - (+50)}{+100 - (+50)}$$

$$(-\tfrac{3}{2}) \times (+50) = \omega_F - 50$$

and $$\omega_F = +50 - 75 = -25$$

or the absolute angular velocity of gear F is 25 r.p.m., counter-clockwise.

It will be well for us to note some of the other angular velocity relationships between the members of the epicyclic train.

The angular velocity of gear F relative to the arm will be

$$-25 - (+50) = -75$$

or F has an angular velocity of 75 r.p.m., in a counter-clockwise direction, relative to the arm.

Gear D has an angular velocity, relative to the arm, which is $+100 - (+50) = +50$ r.p.m., clockwise, or in the same direction as the arm.

This checks with the train value for

$$\frac{\text{Angular velocity of } F \text{ relative to } A}{\text{Angular velocity of } D \text{ relative to } A} = \frac{75}{50} = \frac{3}{2},$$

with opposed senses of rotation, or minus sign of train value.

Another example will direct attention to other possibilities of motion in the uses of this fascinating application of relative motion.

The train shown in Fig. 235 consists of a driver D, a gear I, and an annular F. An arm A rotates about the same fixed axis as D, and carries the pin which serves as axis for I.

The following data are available.

D has 24 teeth and absolute angular velocity of 20 r.p.s., clockwise.

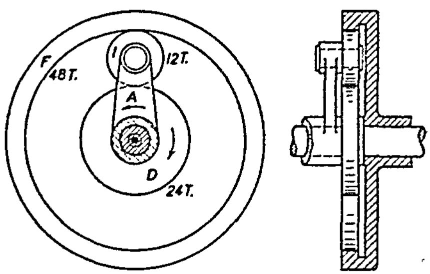


Fig. 235

I has 12 teeth.

F has 48 teeth.

Arm A has absolute angular velocity of 5 r.p.s., counter-clockwise.

We are to find the absolute angular velocity of F.

The train is identified as an epicyclic one because again we note "arm action," or the presence of the rotating axis.

In the identification of an epicyclic gear train, we should note that every gear which has an axis carried by the arm, or is in contact with a gear whose axis is carried by the arm, is a member of the epicyclic train.

The train value is established, again, by noting the value which it would have if the arm were held still.

$$t.v. = -\tfrac{24}{12} \times \tfrac{12}{48} = -\tfrac{1}{2}$$

Then
$$-\frac{1}{2} = \frac{\omega_F - (-5)}{+20 - (-5)}.$$

$\omega_F = (-\tfrac{1}{2} \times 25) - 5 = -17.5$; and the annular has an absolute angular velocity of 17.5 r.p.s., counter-clockwise.

This solution may be checked by reversing the train; that is, by assuming the drive (with velocity of 17.5 r.p.s., counter-clockwise) to be applied at the annular as driver, and determining the absolute angular velocity of D, which is now the follower, under the same conditions of arm motion as before.

Now
$$t.v. = -\tfrac{48}{12} \times \tfrac{12}{24} = -2$$

$$-2 = \frac{\omega_D - (-5)}{-17.5 - (-5)}.$$

$\omega_D = [-2 \times (-12.5)] - 5 = +20$ r.p.s., which is correct for the train operated in this direction of drive, and confirms the relationships we previously derived.

The angular velocity of F relative to the angular velocity of the arm:

$$\omega_{F/A} = \omega_F \rightarrow \omega_A$$

This is a vector difference, and

$$\omega_{F/A} = -17.5 - (-5) = -12.5$$

The angular velocity of D relative to the angular velocity of the arm

$$\omega_{D/A} = \omega_D \rightarrow \omega_A$$
$$= +20 - (-5) = +25$$

Then
$$t.v. = \frac{\omega_{F/A}}{\omega_{D/A}} = \frac{-12.5}{\cdot 25} = -\frac{1}{2}$$

which checks the original data.

The absolute angular velocity of the idler may be determined by identifying an epicyclic train in which D is the driver and I the follower. In this train:

$$t.v. = -\tfrac{24}{12} = -2$$

$$-2 = \frac{\omega_I - (-5)}{20 - (-5)}$$

and $\omega_I = -55$, or 55 r.p.s., counter-clockwise.

As with gear trains in which all axes are fixed, the epicyclic trains may be reverted, and the axis of last and first wheels made coincident.

Figure 236 illustrates such a train, and also introduces another element, the stationary wheel, as one element in an epicyclic train.

In this train, we have the following table of given data.

Gear D has 48 teeth—gear stationary
B has 12 teeth
C has 15 teeth
F has 45 teeth

Arm A makes 100 r.p.m., clockwise, as viewed from the right side.

D, A, and F rotate about coincident axes.

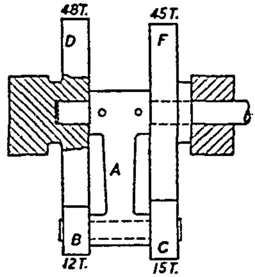


FIG. 236

We are to determine the absolute angular velocity of follower F.

$$t.v. = \tfrac{48}{12} \times \tfrac{15}{45} = +\tfrac{4}{3}$$

$$+\frac{4}{3} = \frac{\omega_F - (+100)}{0 - (+100)}.$$

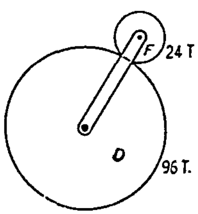
$\omega_F = -133.3 + 100 = -33.3$, or F is turning counter-clockwise as viewed from the right side with speed of 33.3 r.p.m.

It should be noted that the train value is not affected by the stationary nature of the driver. The *train value is dependent only upon the number of teeth*—it is a ratio of angular velocities of driver and follower relative to the arm. The absolute velocity of arm, driver, or follower may be of any value, including the zero value of stationary members, without affecting the train value.

PROBLEMS

Note: In the following problems, if the sense of the angular velocity of the driver is not given, either sense may be assumed. The answer must report whether the sense of the follower is the same as or opposite to that of the driver.

279. The arm of the epicyclic train has an absolute angular velocity of 200 r.p.m., clockwise. Gear D has an absolute angular velocity of 1000 r.p.m., clockwise. Determine the absolute angular velocity of gear F. *Ans.* -3000 r.p.m.



PROB. 279

280. Solve Problem 279, with D changed to a stationary gear.

281. Solve Problem 279 if gear D has an angular velocity of 1000 r.p.m., counter-clockwise.

282. Using the data of Problem 279, except that the angular velocity of the arm is unknown, determine the absolute angular velocity of the arm which will cause F to have zero absolute angular velocity.

283. If the arm velocity is increased beyond the value determined in Problem 282, what is the effect upon the absolute angu-

lar velocity of the follower? What is the effect of decreasing the arm velocity below the value determined in Problem 282?

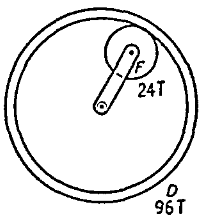
284. The data for the epicyclic train is the same as that given in Problem 279, except that the contact has been changed to internal contact. Determine the absolute angular velocity of F. The arm and annular are mounted upon coincident axes.

Ans. +3400 r.p.m.

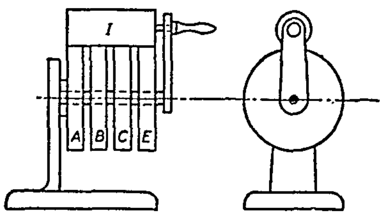
285. Determine the absolute angular velocity of pinion F (Problem 284) if annular D has an absolute angular velocity of 150 r.p.m., clockwise, and the arm has an absolute angular velocity of 200 r.p.m., clockwise.

286. The epicyclic mechanism shown is known as Ferguson's Paradox, and combines a forward, neutral, and reverse drive in one assemblage. It is therefore occasionally employed as the basis of planetary automotive transmissions.

Gears A, B, C, and E are supported on one shaft, which is also the axis of the arm. Gear A is fixed to the frame. B, C, and E may rotate independently. The arm carries a shaft supporting the broad-faced pinion, I. Gear A has 50 teeth, B—49, C—51, and E—50. If the arm makes one revolution, determine the magnitudes and directions of the revolutions of B, C, and E.



PROB. 284



PROB. 286

287. The gear-motor drive shown uses an epicyclic train between the motor and follower shaft. Gear A is cut on the motor shaft and has a speed of 1600 r.p.m.

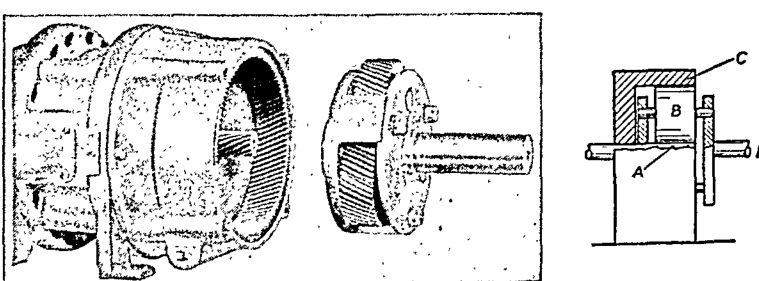
A meshes with B which rotates freely on an axis supported by the arm.

B meshes with C, which is an annular cut in the stationary frame of the motor.

The follower shaft, F, is fastened to the arm.

Gear A has 12 teeth, B—44, and C—100.

Determine (a) the absolute angular velocity of F; (b) the angular velocity of B relative to its own axis. *Ans.* +171 r.p.m.; 390 r.p.m.



Courtesy The General Electric Co.
PROB. 287

288. The gear-motor shown is of the double-reduction type. It consists of a series of two epicyclic trains, each similar to that of Problem 287.

If the motor speed is 1600 r.p.m., determine the absolute speed of the follower shaft. Use the same numbers of teeth as on corresponding gears of Problem 287.

289. The reducing mechanism shown consists of a series arrangement of epicyclic gear trains.

Shaft *D* is driven by a motor whose speed is 1000 r.p.m., and drives *Arm 1*. Gears *B* and *C* are fastened together, and are supported on a shaft carried by *Arm 1*. Gears *A* and *E* are supported by the drive shaft but may rotate independently.

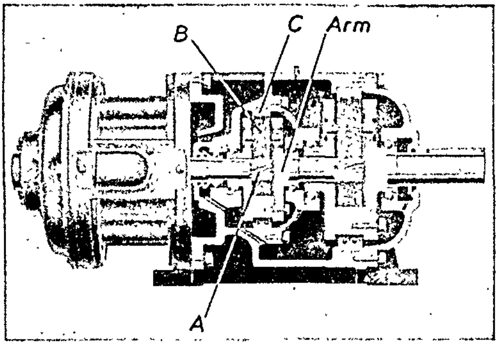
Arm 2 is fastened to gear *E*, and carries the shaft supporting gears *H* and *J*, which are fastened together. Gears *G* and *K* are supported on a shaft whose axis is coincident with that of shaft *D*. *G* and *K* are loose on their shaft.

A drum *F* is fastened to gear *K*.

Gears *A* and *G* are stationary.

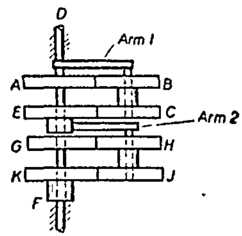
Determine the time required for one revolution of *F*.

A has 104 teeth, *B*—100, *C*—96, *E*—100, *G*—104, *H*—100, *J*—96, *K*—100.



Courtesy The General Electric Co.

PROB. 288



PROB. 289

290. A governing mechanism is of the form shown.

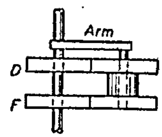
The arm velocity may be varied in magnitude, but is constant in sense. The driver, *D*, has an absolute angular speed of 100 r.p.m.

Design the train to fulfill the following conditions. All gears are to be of the same diametral pitch.

(a) For absolute arm speeds below 500 r.p.m., the follower, *F*, is to have an absolute angular velocity of one sense.

(b) For absolute arm speeds above 500 r.p.m. the follower, *F*, is to have an absolute angular velocity of reverse sense from that of part (a).

291. Design an epicyclic gear train for a reduction of speed of $-440 : 1$. The axes of driving gear and follower gear are to be coincident. Use 4 gears. The arm rotates at an absolute speed of 50 r.p.m. Only those gears given in the manufacturer's list of stock sizes (see Problem 268) are to be used.



PROB. 290

292. Solve Problem 291, with a reduction of speed $= +260 : 1$.

293. Shaft *D* has an absolute speed of 960 r.p.m. Gears *A* and *C* are keyed to *D*.

Gear *B* drives the arm of the epicyclic train. Gear *E* has external teeth meshing with *C*, and internal teeth meshing with *G*.

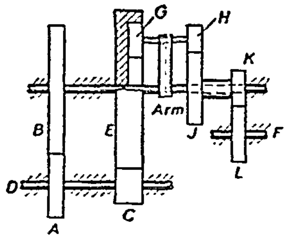
G and *H* are both keyed to a shaft carried by the arm.

J is the follower of the epicyclic gear train, and is fastened to K.

L is keyed to the follower shaft F.

Determine the absolute angular velocity of F.

A has 24 teeth; B—48; C—36; E—external—108, E—internal—96; G—24; H—12; J—36; K—16; L—34.



PROB. 293

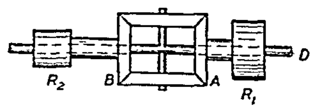
294. Reels R_1 and R_2 of a textile machine are fastened to gears A and B, respectively.

Shaft D drives the arm of the bevel epicyclic train.

The absolute linear speed of the surface of R_1 is 240 f.p.m., and its diameter is 12 in.

R_2 has a diameter of 6 in.

R_1 and R_2 rotate with the same sense of absolute angular velocity.



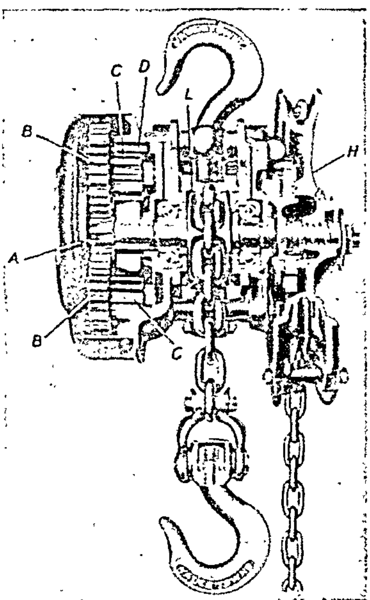
PROB. 294

(a) At what speed must shaft D be driven for R_2 to have an absolute surface speed of 160 f.p.m.?

(b) Reel R_1 is to rotate three times as fast as reel R_2 in the opposed sense. Determine the required absolute angular velocity of D for an absolute speed of R_2 = 100 r.p.m.

295. The hoist shown is operated by a hand-wheel, H, which turns gear A. B and C are fastened together, and their axis is carried by the arm which is fastened to the load sheave, L. B meshes with A, and C with annular D, which is stationary. If the pitch diameters of H and L are 12 in. and 3 in. respectively, what is the mechanical advantage?

Gear A has 12 teeth, B—29, C—11, and D—42.

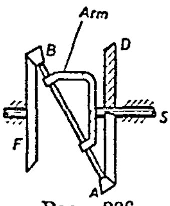


Courtesy The Yale & Towne Mfg. Co.
PROB. 295

296. The arm is fastened to shaft S and carries pinions A and B.

Gear D is stationary.

Determine the absolute angular speed of F if the speed of S is 630 r.p.m.



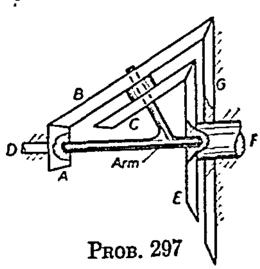
PROB. 296

Gear D has 78 teeth; A—18; B—20; and F—84.

Ans. 20 r.p.m.

297. The bevel epicyclic train shown is known as Humpage's gear.

Shafts D and F have coincident axes. A meshes with B, and C with E. B and C are fastened together and are carried by the arm which is supported on the same axis as D and F, but may rotate independently.



PROB. 297

Gear A is fastened to shaft D, and E is fastened to shaft F.

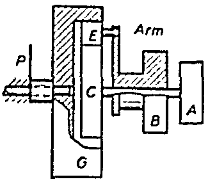
Gear G is stationary, and is in mesh with B.

If D rotates at 600 r.p.m., find the absolute angular velocity of F.

A has 12 teeth; B—48; C—20; E—36; G—72.

298. The mechanism of a meter for indicating the synchronized speed of two turbines is shown.

One turbine is belted to pulley A, which drives gear C.



PROB. 298

The other turbine is belted to pulley B, which drives the arm of the epicyclic gear train.

Pinion E, carried by the arm, meshes with annular G, and with C.

The pointer, P, is fastened to G.

The diameter of pulley A is $\frac{1}{3}$ that of B.

Select suitable numbers of teeth for gears C, E, and G, if the pointer is to remain stationary when both turbine belts have the same linear velocity.

299. If gear C of Problem 298 has 24 teeth, and annular G has 56 teeth, determine the ratio of the diameters of pulleys A and B if pointer P remains stationary when the turbine belts have the same absolute speed. *Ans.* 3:10.

300. The machine drive shown consists of a series of epicyclic gear trains.

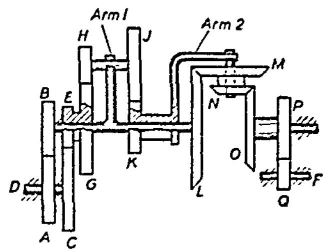
Gears A and C are keyed to the drive shaft D.

B is keyed to the shaft of *Arm 1*, which is also keyed to gear L.

Gears E and G are fastened together, and supported freely on the arm shaft, as are gear K and *Arm 2*.

H and J are carried by *Arm 1*, and are fastened together.

M and N are fastened together, and carried by *Arm 2*.



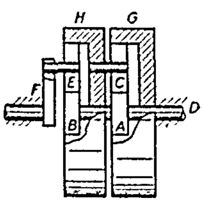
PROB. 300

O and P are fastened together.

P drives Q which is keyed to the follower shaft.

The absolute speed of shaft D is 100 r.p.m.

Find the absolute angular velocity of F. Gear A has 32 teeth; B—24; C—36; E—18; G—64; H—28; J—56; K—32; L—54; M—24; N—12; O—36; P—36; Q—12.



PROB. 301

301. A selective planetary automotive transmission is shown in the diagram.

The engine shaft D is keyed to gears A and B.

Pinion C meshes with gear A and annular G, and floats freely on a pin carried by annular H.

Pinion E meshes with gear B and annular H, and floats freely on a pin carried by arm F, which is keyed to the propeller shaft.

Two brake bands (not shown) make it possible to lock either annular G or H.

If the drive shaft rotates at 2000 r.p.m., determine the speed of the propeller shaft when

(a) G is stationary,

(b) H is stationary.

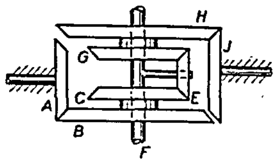
Gears A and B have 20 teeth each, and annulars G and H have 70 teeth each.

302. Design an additional step for the transmission of Problem 301 so that the propeller

shaft will rotate in the opposite direction from the drive shaft. Also report the resulting angular speed of F.

303. The ship indicator shown records differences of speed between a port and a starboard engine which have opposite senses of rotation.

Gear A is driven by the port engine; J by the starboard engine.
Gears B and C are fastened together, as are gears G and H.
B and C, and G and H, are supported loosely on shaft F.
Gear E is carried by the arm.
Determine the angular velocity of F when:
(a) Both engines have the same speed
(b) Port engine—175 r.p.m.
Starboard engine—140 r.p.m.
(c) Port engine—140 r.p.m.
Starboard engine—175 r.p.m.



Prob. 303

Gears A and J have 24 teeth each; B and H—56; C and G—40; E—20.

88. Rigid Connectors. Link-Work. A careful investigation of link-work will reveal that while the subject is extensive, a well-defined foundation is available. This basis will serve to give direction to such efforts—to orient methods of attack, and make it possible to analyze the kinematics of these bodies without confusion, and without the disjointed effort attendant upon treating each mechanism as a new problem.

The *four-bar linkage*, or quadric mechanism, is of primary importance.

In simplest physical form each four-bar linkage consists, as the name suggests, of a series of four rigid bodies. It is inevitable that we should reach this objective in a search for a mechanism of such general nature that its properties might be common to all mechanisms. A series of three bodies, when pinned together, becomes incapable of offering relative motion between the three bodies, or links. Such a combination is suggested in Fig. 237. We note there that no flexibility for relative motion is presented, for three lines when joined at their ends give but one triangle, and the sides when pinned together cannot move relative to each other. When three links are thus joined, and one is fixed, all links have been fixed.

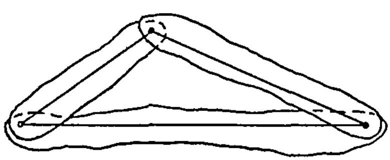


Fig. 237

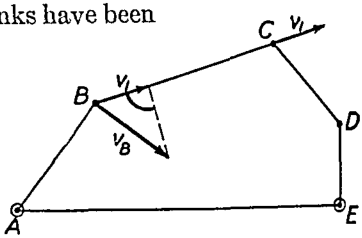


Fig. 238

In the case of five or more links (see Fig. 238) there is an over-abundance of possibilities of relative motion. If one link, as AE, is fixed and another, like AB, permitted to move, BC, CD, and DE will move but they will move

in unpredictable fashion. If, for example, point B is given a linear velocity v_B, the orthogonal component v_1 is transmitted to point C as its orthogonal component in the direction BC. One orthogonal component is ineffective when it is the only information available, and when the inclination of C's resultant velocity is unknown. This series of five links, then, possesses motion possibilities but has no motion certainties throughout. Unless some external constraint is added to give direction to C's velocity its resultant motion cannot be predicted. A series of five or more links is therefore an example of indeterminate motion. A series of four links must be the basic series, yielding not only relative motion between links but also a definite or predictable form of motion. As the fundamental element of mechanism, this four-bar linkage has very penetrating value in the development of methods of investigation of all mechanisms.

All mechanisms whose motion is determinate throughout must have, as their kinematic background, the equivalent of a series of four links or of multiple arrangements of four-link series.

The series of four links is illustrated in Fig. 239. If one link, as AD, be fixed so that it has no absolute motion, and point B be given a linear velocity v_B, the orthogonal component v_1 is again transmitted along rigid body BC, and becomes C's orthogonal component in that direction. In this case, however, the inclination of C's resultant velocity is known, for C is constrained by link CD so that it must always move in a circular path about D, and it must have an inclination of velocity at any instant which is perpendicular to CD. Point C has one known orthogonal component and a known inclination of velocity and it satisfies the demands of Theorem I.

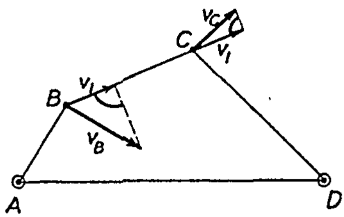


FIG. 239

This investigation of the velocity properties of B and C shows that this mechanism is completely determinate; with the several weapons of attack which have been developed, such as the instantaneous axis of velocities, or the components of translation and rotation, the velocity of any point on body BC may be determined. Bodies AB and CD are examples of pure rotation. Since the resultant velocity of a point on each is known, the velocity of any point is determinate. AD is fixed, and every point on that body has zero absolute velocity.

The four-bar linkage, as such, is frequently used as a mechanism. Its most customary aspect is shown in Fig. 240 with link AD fixed on the frame of a machine. The line AD is known as the *line of centers.* AB is called a *crank,* which is defined as a body having pure rotation about an axis lying at one end of the line of centers. CD is also a crank, since it con-

forms to the kinematical definition just given. The fourth member, BC, joins the moving ends of the two cranks, and is called the *connecting rod.*

In use, one crank is made the driver, and the other crank becomes the follower. The driving crank usually rotates at constant speed, and the fol-

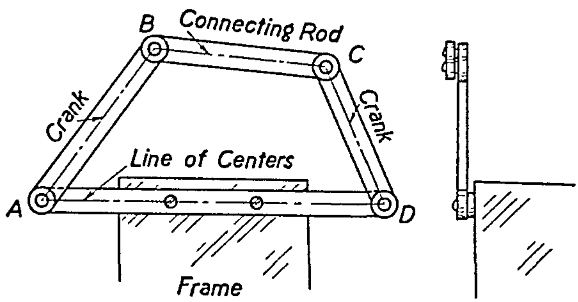


FIG. 240

lower's motion becomes constant or variable depending entirely on the relative proportions of the four links. The follower may either oscillate, or, in the case of the drag-link mechanism, make complete revolutions.

In the previous analyses of velocity problems, the vectors have been used as tools of analysis. As we pursue our study of mechanisms upon the basis of four-bar linkages, this method will be found somewhat cumbersome. When, as in the past, interest centers in the linkages' kinematical properties for but one instantaneous position the vector solution is adequate and efficient. However, in a study of such properties for many positions, the vector solution, while still adequate, is now inefficient, because the graphical resolutions must be repeated, in complete detail, a great number of times.

It will be efficient to substitute a graphical method which will yield results which, both as to direction and magnitude, shall have equivalent validity but which will involve far less effort in the graphical constructions.

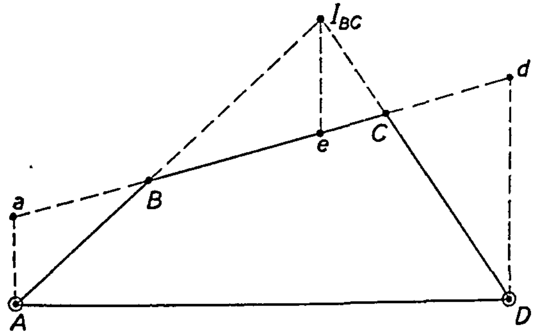


FIG. 241

Figure 241 shows a four-bar linkage of known dimensions, $ABCD$. The given data also includes the angular velocity of the driving crank AB.

The instantaneous axis of connecting rod BC lies at I_{BC}.

If, from point A, a line Aa and from I and D lines $I_{BC}e$ and Dd, be drawn, both parallel to Aa, we have a basis of method.

$$L.S.B. = \omega_{AB} \times AB$$

where $L.S.B.$ is the linear speed of point B

and $L.S.C. = \omega_{CD} \times CD$

where $L.S.C.$ is the linear speed of point C.

Then

$$\frac{L.S.C.}{L.S.B.} = \frac{\omega_{CD} \times CD}{\omega_{AB} \times AB}$$

but, from the properties of the instantaneous axis,

$$\frac{L.S.C.}{L.S.B.} = \frac{I_{BC}C}{I_{BC}B}.$$

Now triangle ABa is similar to triangle $I_{BC}Be$, and triangle DCd is similar to triangle $I_{BC}Ce$.

Aligning the corresponding sides of these triangles in correct proportion, it is found that

$$\frac{CD}{Dd} = \frac{I_{BC}C}{I_{BC}e}$$

and

$$\frac{Aa}{AB} = \frac{I_{BC}e}{I_{BC}B}.$$

Now, multiplying,

$$\frac{CD}{Dd} \times \frac{Aa}{AB} = \frac{I_{BC}C}{I_{BC}B} = \frac{L.S.C.}{L.S.B.}$$
$$= \frac{\omega_{CD} \times CD}{\omega_{AB} \times AB}.$$

Simplifying,

$$\frac{\omega_{CD}}{\omega_{AB}} = \frac{Aa}{Dd}.$$

Then the *speed ratio of the two cranks is equal to the inverse ratio of the lengths of any two parallel lines, drawn from the fixed ends of the cranks to intersect the connecting rod.*

In application a further simplification presents itself. The derivation of a speed ratio basis has rested upon two lines Aa and Dd, which must satisfy but one requirement; that is, they must be parallel to each other. Any pair of parallel lines originating at the fixed axes may be used. Then the crank AB, already drawn in each position of the mechanism, may be used as one

parallel, and a line Dd, drawn from D parallel to AB, as in Fig. 242, will yield the measurement of distance needed to completely establish the speed ratio. Or CD may be used as one line, and a line Aa, parallel to CD at fixed axis A, may be used and will yield the equivalent ratio.

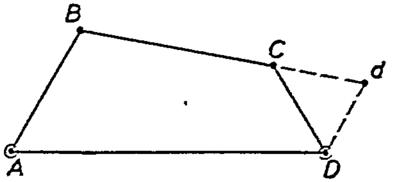


Fig. 242

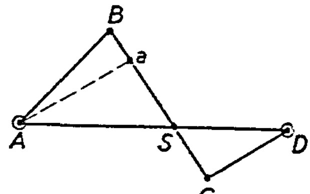


Fig. 243

Another possible solution is suggested by Fig. 243. When the four-bar linkage is in the position shown, the connecting rod crosses the line of centers and divides that line into segments which are related so as to yield the desired speed ratio without further graphical construction.

The speed ratio

$$\frac{\omega_{CD}}{\omega_{AB}} = \frac{Aa}{CD}.$$

In similar triangles ASa and DSC, Aa and As are sides corresponding to CD and DS, respectively.

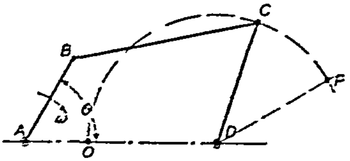
Then

$$\frac{\omega_{CD}}{\omega_{AB}} = \frac{Aa}{CD} = \frac{AS}{SD}.$$

These are all equivalent geometrical devices for obtaining the speed ratio, and a complete investigation of the four-bar linkage throughout its cycle of motion may employ in some positions one device, while in another position (as when the connecting rod crosses the line of centers) another method may be most direct.

PROBLEMS

304. Determine the speed ratio of the cranks when $\theta = 30°$.



Prob. 304

$AB = 3.0$ in.; $BC = 6.0$ in.; $CD = 4.0$ in.; $AD = 6$ in. *Ans.* 0.66

305. Solve Problem 304, with θ changed to 47°.

306. Solve Problem 304, with θ changed to 35°.

307. Using the linkage of Problem 304, with AB as driving crank, determine the angle through which CD oscillates for one revolution of AB.

308. Determine the time ratio of *forward stroke* of crank CD, in Problem 307 *to return stroke,* if AB has constant velocity.

309. If the angular velocity of AB, Problem 304, is 1 radian per sec., clockwise, plot a curve giving the angular velocities of CD as ordinates, and time intervals as abscissae, for a complete revolution of AB.

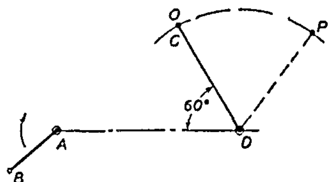
310. Redesign the linkage of Problem 304, by modifying the length of driving crank AB and connecting rod BC, so that CD will oscillate between points O and P. Angle $ODP = 150°$.

Determine the time ratio of $\frac{\text{Forward stroke}}{\text{Return stroke}}$ of crank CD, *if* AB *rotates at constant* angular velocity.

311. Solve Problem 309, plotting a curve showing the angular-displacement-time relationship for CD, and graphically differentiating to obtain the curve of angular-velocity-time relationship.

312. Two shafts, A and D, 6 in. apart are to be connected by a four-bar linkage. Crank AB, 2 in. long, rotates clockwise with constant speed.

CD is 4 in. long. Crank CD is to oscillate between positions OD and PD. Determine the length of the connecting rod BC, the angle of oscillation of crank CD, and the time ratio of $\frac{\text{Forward stroke}}{\text{Return stroke}}$.



PROB. 312

89. The Four-Bar Linkage as Equivalent Fundamental Mechanism. Determinate motion in the case of connected bodies has been reduced in the previous article to a fundamental form—the four-bar linkage. It follows that all mechanisms are kinematically equivalent to a single four-bar linkage, or to multiple arrangements containing series of these fundamental linkages. It is possible to analyze the motion of any mechanism by identifying the equivalent four-bar linkage.

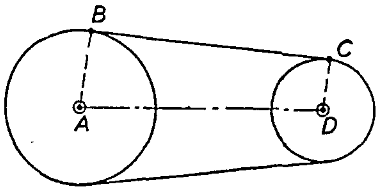


FIG. 244

For example, the pulleys shown in Fig. 244 are connected by an open belt. An equivalent four-bar linkage is identified by finding four bodies which conform to the definitions of line of centers, cranks, and connecting rod, respectively.

AD, the line joining the fixed axes of the pulleys, is evidently the line of centers. A radius AB of the left pulley is a crank, for it is a body moving with pure rotation about a fixed axis lying at one end of the line of centers. Radius CD is likewise a crank. The portion, BC, of the belt behaves, in its transmission of motion, exactly like the rigid connector of a four-bar linkage and is therefore the connecting rod.

Having identified the four-bar linkage, the speed ratio may now be determined. In four-bar linkage $ABCD$,

$$\frac{\omega_{CD}}{\omega_{AB}} = \frac{AB}{CD}$$

for AB and CD are parallel lines from the fixed axes which intersect the connecting rod.

This speed ratio is valid because it agrees with the speed ratio of belt and pulley, where

$$\frac{\omega_{CD}}{\omega_{AB}} = \frac{R_{AB}}{R_{CD}}$$

where R_{AB} is the radius of the left pulley and R_{CD} is the radius of the right pulley.

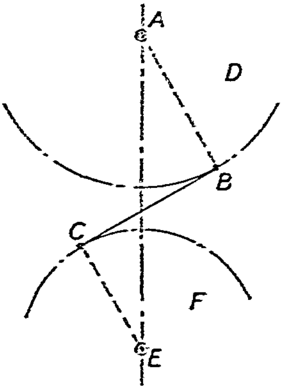


FIG. 245

Involute gears furnish a second example in which the action of the equivalent four-bar linkage as fundamental background may be observed.

We find in Fig. 245 a pair of involute gears, shown in skeletonized form as base circles connected by pressure line BC.

As gears, the speed ratio

$$\frac{\omega_F}{\omega_D} = \frac{D_{B.D.}}{D_{B.F.}}$$

where $D_{B.D.}$ is the diameter of the base circle of the driver, and $D_{B.F.}$ is the diameter of the base circle of the follower.

The members of the equivalent four-bar linkage may be identified by checking each member against the definition of the link it represents.

AE is a line of centers
AB is a driving crank
CE is the follower crank
BC is a connecting rod.

Then the speed ratio is

$$\frac{\omega_F}{\omega_D} = \frac{AB}{EC}$$

for AB and EC are the parallel lines from the fixed axes which determine the speed ratio.

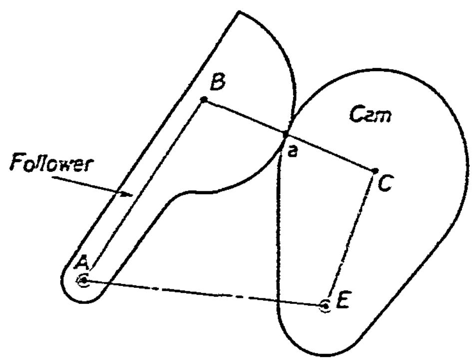


FIG. 246

In Fig. 246, a cam is shown in contact with a rotating follower.

The cam and follower are in contact at point a. Point C is the center of curvature of the cam surface at the point of contact, and point B is the center of curvature of the follower surface at the point of contact. Line BaC must be a straight line, for when two curved surfaces are in contact they have but one common normal.

Point B, a point on the follower, may rotate only in a circular path about A, and point C, on the cam, is constrained to move in a circular path about E.

Then the equivalent four-bar linkage comprises:

AE = line of centers
EC = driving crank
BC = connecting rod
AB = follower crank.

If links of the dimensions of these four members are substituted for the cam and follower, the resulting mechanism will reproduce the kinematic properties of the original pair.

It should be noted that when the centers of curvature change their location with changes in the character of the surfaces of cam and follower, the relative sizes of the four links of each equivalent linkage may change. In each case, it will be necessary to establish the location of the centers of curvature before identifying the equivalent four-bar linkage.

These reductions of mechanisms to their equivalent four-bar linkage serve to supplement a knowledge of speed relationships particularly in special mechanisms because one common basis continually presents itself to coordinate their study. The four-bar linkage thus serves as a method of attack upon velocity, and, later upon acceleration, particularly in the case of problems which may be sufficiently unorthodox to yield to no other readily available means of solution.

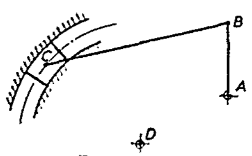
Linkages are of interest, however, not alone as a basis of attack upon mechanism problems by substitution of the equivalent linkage, but, in their own right, as an important group of widely used mechanisms.

PROBLEMS

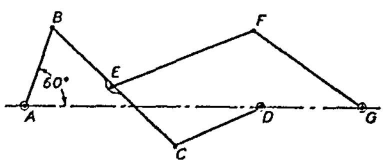
313. Identify the four members of the equivalent four-bar linkage.

AB = 2.4 in.; BC = 6.2 in.; CD = 4.0 in.; AD = 3.3 in. D is the center of curvature of the fixed circular slot. Point A is fixed.

If the angular velocity of AB is 1000 r.p.m., clockwise, determine the angular velocity of equivalent link CD when angle DAB = 120°.



PROB. 313



PROB. 314

314. Identify a series of three equivalent four-bar linkages.

If AB has an angular velocity of 250 r.p.m., counter-clockwise, determine the angular velocities of CD and FG for the position shown.

BEC is a continuous member. $AB = 2.7$ in.; $BC = 5.0$ in.; $CD = 3.0$ in.; $AD = 7.8$ in.; $CE = 2.8$ in.; $EF = 5.0$ in.; $FG = 4.4$ in.; $DG = 3.3$ in.

Axes *A*, *D*, and *G* are fixed.

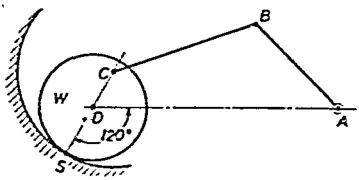
315. Check the results of Problem 314 by resolution of velocity vectors.

316. Wheel *W* is in pure rolling contact at *S* with a curved track, which is fixed.

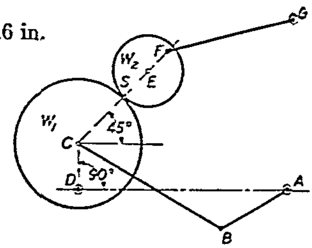
AB has an angular velocity of 120 r.p.m., clockwise.

Identify the equivalent four-bar linkage, and determine the angular velocity of *W*, in the position shown. *D* is the center of the wheel, which has a diameter of 3.3 in.

$AB = 3.7$ in.; $BC = 4.6$ in.; $CD = 1.2$ in.; $AD = 7.6$ in.



Prob. 316



Prob. 317

317. Wheels W_1 and W_2 are in pure rolling contact at *S*. The driving crank, *AB*, has an angular velocity of 1 radian per sec., clockwise.

Diameter of $W_1 = 4.0$ in., diameter of $W_2 = 2.2$ in.

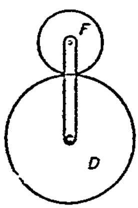
$AB = 2.6$ in.; $BC = 5.3$ in.; $CD = 1.4$ in.; $AD = 6.8$ in.; $EF = 0.9$ in.; $FG = 4.3$ in.; $AG = 5.4$ in.

A, *D*, and *G* are fixed axes.

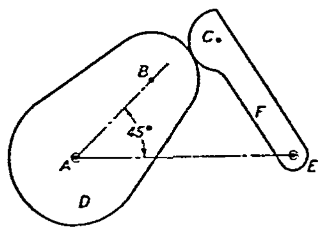
C is the center of W_1, and *E* is the center of W_2.

Identify the background series of equivalent four-bar linkages, and determine the angular velocity of *FG*.

318. In the epicyclic train shown, the wheels are in pure rolling contact. *F* has a diameter of 2.0 in., and *D* has a diameter of 4.0 in.



Prob. 318



Prob. 319

The absolute angular velocity of *D* is 1 radian per sec., clockwise.

The absolute angular velocity of the arm is $1\frac{1}{3}$ radians per sec., clockwise.

Identify the equivalent four-bar linkage.

319. The cam *D* drives follower *F*. *D* has an angular velocity of 30 r.p.m., clockwise. Identify the equivalent four-bar linkage, and determine the angular velocity of *F* in the position shown.

$AB = 3.4$ in.; $CE = 4.4$ in.; $AE = 7.0$ in. *B* is the center of curvature of the cam face at the point of contact, and the radius of curvature of the cam face at that point is 1.5 in.

C is the center of curvature of the face of the follower at the point of contact, and the radius of curvature is 1.0 in.

90. Relative Size of Links. The four-bar linkage, comprising a machine frame serving as line of centers, cranks which are usually steel bars but which may be wheels, and a steel bar serving as connecting rod—all members being pin-connected—presents an easily manufactured and simple means of providing a wide range of follower motions.

Flexibility in the nature of motion transmitted to the follower depends upon the relative sizes of the four links.

An important group of linkages serves to produce special forms of motion when particularized displacement is of primary interest, and modification of velocity is secondary. Applications of this nature are found where it is desired that a point or cutting tool trace out a particular curve, such as a circle, a straight line, or an ellipse. When the force involved is small, the linkage is an instrument; when great, it becomes a machine. Some examples of such displacement-determining linkages are discussed in the following articles.

91. Parallel Motion Linkages. A four-bar linkage having equal and parallel cranks, with the connecting rod and line of centers parallel, is illustrated in Fig. 247. Here we note that, since AB and CD are always equal as well as parallel, the speed ratio is constant and equal to 1 : 1.

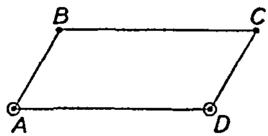


Fig. 247

An example of such a linkage is the side- or parallel-rod of a locomotive. The engine is connected to one pair of the driving wheels, and the motion of this pair is transmitted to other pairs through the side rods.

Drafting machines which combine the functions of T-square and triangle make use of the equal and parallel crank linkage, as shown in Fig. 248.

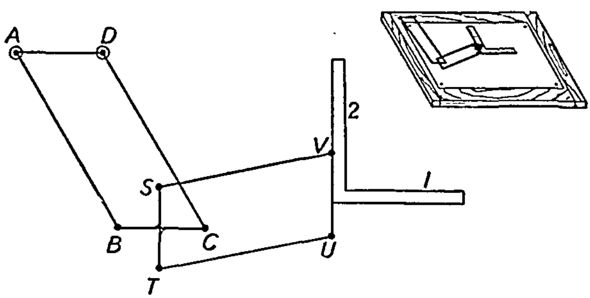


Fig. 248

AB and CD are equal and parallel, and line of centers AD is locked to the drawing board in a horizontal position. Then connecting rod BC will remain horizontal as the cranks are turned. Linkage $STUV$ is also an

equal and parallel crank linkage, with line ST fixed at right angles to BC. Then UV will remain vertical in any position of the cranks. The blades *1* and *2* will remain horizontal and vertical, respectively, as they are moved over the drawing surface.

Another parallel motion linkage is the pantograph. This is discussed in Art. 97.

92. Straight Line Linkages. The relative size of the four members of a four-bar linkage has already been referred to as the determining factor in fixing the nature of motion transmitted from driver to follower.

Many forms of linkages have been devised through assignment of proper relative size to the individual links, so that a straight line will be generated by a point of the linkage. The term "generated" when used in this sense means that a previously existing line is not used as template or guide, as in the pantograph, but that the linkage, through its inherent properties, will trace the required line.

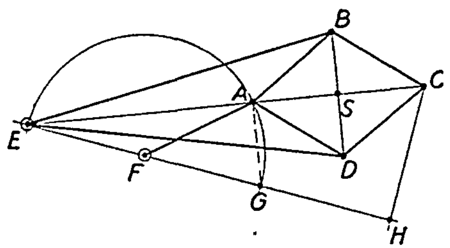


Fig. 249

The *Peaucellier's Cell* is a linkage designed as in Fig. 249. The cell or pinned rhombus $ABCD$ is composed of four equal links. Pins B and D are joined to a fixed axis E by cranks BE and DE, which are of equal length, and pin A is joined to fixed axis F by crank AF, equal in length to the line of centers EF.

A tracing point or tool placed at pin C will trace a line CH which is constantly perpendicular to the line of centers.

The construction of the figure shows that in any position of the linkage, line EAC is one straight line. Moreover, the diagonals BD and AC bisect each other at right angles at point S.

Then the product

$$\begin{aligned} EA \times EC &= (ES - SA)\,(ES + SC) \\ &= \overline{ES}^2 - \overline{SA}^2 \\ &= (\overline{EB}^2 - \overline{BS}^2) - (\overline{AB}^2 - \overline{BS}^2) \\ &= \overline{EB}^2 - \overline{BS}^2 - \overline{AB}^2 + \overline{BS}^2 \\ &= \overline{EB}^2 - \overline{AB}^2. \end{aligned}$$

But EB is of constant length, as is AB. Then the product $EA \times EC$ is a constant in any position of the linkage.

If now the circular path of point A is drawn to meet the line of centers at point G, it will be noted that angle EAG is inscribed in a semi-circle and is therefore a right angle.

If in any position of the linkage a perpendicular from point C is drawn to meet the line of centers at H, angle CHE is also a right angle. Then triangles ECH and EAG are similar triangles, and,

$$\frac{EC}{EG} = \frac{EH}{EA}.$$

It follows that

$$EG \times EH = EC \times EA = \text{a constant.}$$

But EG ($= 2\ EF$) is already constant.

Therefore EH is constant, and point C has a locus which is line CH, perpendicular at all times to EF, and, consequently, a straight line.

Peaucellier's cell linkage therefore generates a line which is a mathematical or true straight line.

Other pin-connected linkages may be used to generate a line which is a close approximation to a straight line. An example of these is that due to *Watt*, which is illustrated in Fig. 250.

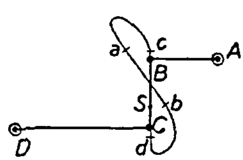


FIG. 250

Two cranks, AB and CD, are mounted upon fixed axes. A point, S, of the connecting rod will trace a path which, as shown in the figure, is a double-looped curve, and two portions ab and cd of the curve will closely approximate straight lines. The location of the point S whose path-segments most nearly approximate the straight line is such that distances SB and SC are inversely proportional to the lengths of the nearest cranks, AB and CD, respectively.

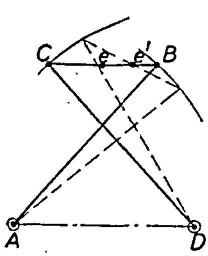


FIG. 251

Another example of approximate straight-line motion is that of *Tchebicheff*, which is illustrated in Fig. 251. The cranks AB and CD are of equal length and are crossed, and the connecting rod BC in its mean position is parallel to the line of centers. The mid-point, e, of the connecting rod will trace an approximate straight line parallel to the line of centers.

The foregoing examples of pin-connected linkages have been of chief interest when a particular path of motion, or displacement, is the basic requirement.

Other linkages whose design is also intended to produce particularized paths of motion will be discussed with the subject of sliding linkages.

93. Design of Linkages for Specific Velocity Ratio. By design of relative size of links, we may also produce linkages in which the modification of velocity is of first concern.

Drag Link. The drag link is shown in Fig. 252, and the curve of Fig. 253 reports the velocity of the crank, CD, which is the follower, for a constant velocity of the crank AB, serving as driver. The name *drag link* should be

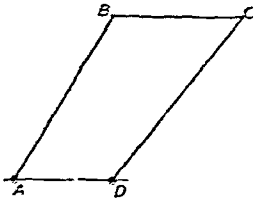


Fig. 252

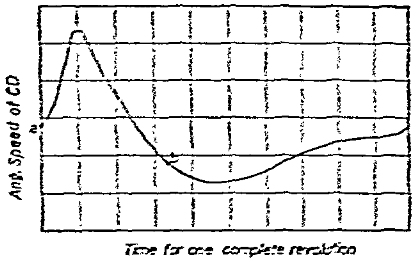


Fig. 253

reserved for four-bar linkages in which both cranks make complete revolutions. This is accomplished by proportioning the relative sizes of the four members, so that there shall be no "dead-point" positions.

These proportions follow:

$$BC > AD + CD - AB$$
$$BC < AB + CD - AD$$

The resulting four-bar linkage presents an interesting approach to the problem of "quick return."

If a man operates a hand-saw so that one-half of his time is spent in cutting and one-half in returning the blade to start a fresh cutting stroke, then one-half of his time might be considered waste if the measure of productive work is that time when the blade is actually cutting. Some waste is of course necessary, for the tool must be returned after each cutting stroke. The over-all efficiency of the job may be improved by insisting that during the stroke used for returning the blade he work at increased speed, leaving a greater portion of his total time for cutting strokes.

In the case of the human example this might be unkind, and difficult to accomplish. When a reciprocating machine saw is substituted, the mechanisms may be so designed that the saw blade is travelling at more rapid speed during the return stroke than during the cutting stroke.

In the drag link example, it will be noted that during the portion of time represented by the curve which lies between points a and b (Fig. 253) the follower crank is travelling rapidly, while during the balance of the stroke it has a slower and more uniform speed. If, then, the linkage is connected

with a cutting tool, as in Fig. 254, and the high-speed portion of CD's motion is made the return stroke, the balance being assigned to cutting stroke, the blade will cut for $\frac{2}{3}$ of the time given to one revolution and return to its starting position for a fresh cut in $\frac{1}{3}$ of this time. Then $\frac{1}{3}$ of the total time the machine is in operation is devoted to return strokes, and $\frac{2}{3}$ is being spent in cutting.

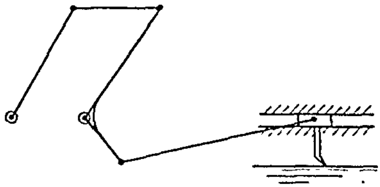


FIG. 254

A practical application of this drag link mechanism is found in the illustration of Fig. 255 which illustrates the modification of a four-bar linkage to different physical form, but with equivalent motion properties.

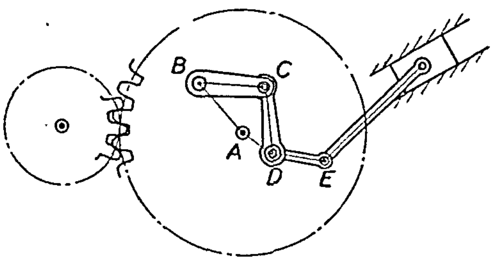


FIG. 255

The mechanism illustrated is a *Dill Slotter* which has been frequently employed in machine tools.

The lines representing the equivalent drag-link linkage are AD = line of centers, CD = crank, BC = connecting rod, and AB = crank.

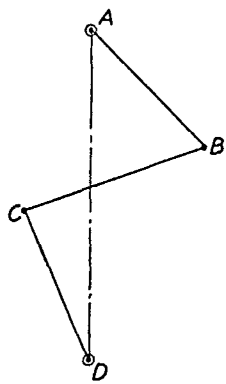


FIG. 256

The large gear turns about fixed axis A, carrying with it the pin B, and therefore equalling the action of a crank AB. A connecting link BC serves as connecting rod, and causes the arm CD to move about center D which is an axis supported in bearings on the machine frame. As arm CD rotates the arm DE, which is fixed to CD, will rotate at the same angular velocity. A connecting rod, pinned to DE at E, drives the ram which carries the cutting tool of the slotter.

The Non-Parallel Equal Crank Linkage is illustrated in Fig. 256, and the curve of Fig. 257 shows the type of speed ratio which is developed This type of speed ratio, with a larger portion of the total time per revolution devoted to fairly uniform speed ratio, and a smaller portion devoted to a rapid motion of the follower, presents another basis for a quick-return drive.

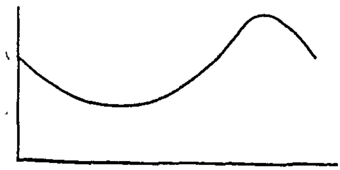


FIG. 257

As the possibilities of the linkage itself are analyzed, however, it will be

found that, as in many linkage applications, the problem of a *dead-point* position arises. The significance of dead-point positions may be noted by considering the ordinary crank-and-connecting-rod mechanism shown in Fig. 258. If the sliding block is the driver, as it would be in a steam or internal combustion engine, the pressure P on the piston-head will cause the crank AB to rotate until it reaches the position where AB and BC are in line (B at B')—the dead-point position.

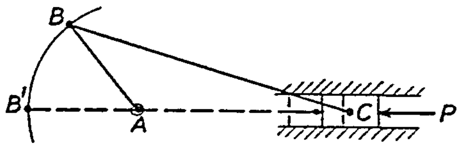


Fig. 258

The connecting rod BC will now tend to stretch the crank AB but has no turning effect. In engines, reservoirs of energy, like fly-wheels, are employed to carry the linkage beyond the dead-point positions.

A kinematic device, the *centrode,* may be employed, and the non-parallel equal-crank linkage offers an opportunity for such an application. Here, too, the crank and connecting rod will line up and yield a dead-point position.

If the line AB of Fig. 259 is made a line of centers, then CD is the connecting rod. $AB = CD$, and $AD = BC$. In the position shown the instantaneous axis of velocities of DC is I_{CD} at the intersection of the two cranks. If we plot the locus of the instantaneous axes of CD throughout one complete revolution, we find that this locus is a continuous curve, and that the curve is an ellipse with foci at points A and B.

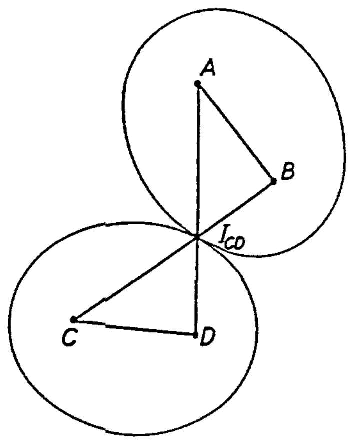


Fig. 259

Now if CD is made a line of centers and AB allowed to make one complete revolution, the locus of its instantaneous axis will be another and equal ellipse, in this case having foci at C and D.

These two ellipses may be substituted for the original non-parallel equal crank linkage by allowing them to roll on each other, with foci A and D serving as fixed axes. The action of rolling ellipses has already been discussed (Art. 58). To secure positive motion throughout the complete revolution, teeth may be cut, using the rolling ellipse as pitch ellipse. The motion of such elliptical gears is equivalent to that of the linkage $ABCD$ which has been replaced, and the speed ratio between ellipses is the same as that obtained by analyzing the crank motion. Since the centrode substitutes develop

no dead-point positions, they have an advantage over the original linkage.

Other centrode substitutions may be made than in the case of the non-parallel equal-crank linkage to overcome dead-point positions, but in many cases the centrodes obtained by plotting the locus of instantaneous axis positions are not continuous curves. The linkage may be supplanted, however, by small portions of the centrode which serve to carry the motion by the dead points.

PROBLEMS

320. The drag-link mechanism shown is to be used for a quick-return motion. Crank AB has constant angular velocity.

Investigate its properties, reporting:

(a) A curve of the angular velocity-time relationship for CD.

(b) The regions of the curve which should be used for forward and return strokes, and their time ratio.

$AB = 3.4$ in.; $BC = 3.0$ in.; $CD = 4.0$ in.; $AD = 1.6$ in.

321. If CD of Problem 320 is to make complete revolutions, what is the maximum length to which AD may be increased? (All other dimensions remain fixed.)

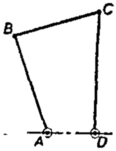
322. Redesign the mechanism of Problem 320 by changing the distance AD to 2.6 in., and determining the minimum permissible length of connecting rod if CD is to make complete revolutions. (All other dimensions remain fixed.)

323. Using the data of Problem 322, determine the maximum length of connecting rod if CD is to make complete revolutions.

324. Design the centrodes which may be substituted for the linkage shown. $AB = 2.0$ in.; $BC = 4.0$ in.; $CD = 2.0$ in.; $AD = 4.0$ in.

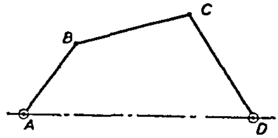
325. Design the portion of centrode of BC, with AD fixed, and of AD with BC fixed, between the limiting positions when AB and BC are in line.

$AB = 3.0$ in.; $BC = 4.0$ in.; $CD = 4.0$ in.; $AD = 7.6$ in.



PROB. 320

PROB. 324



PROB. 325

94. Link-Work Involving Sliding. The four-bar linkage which has been observed in its role of fundamental mechanism background, or as practical application, has been pin-connected. Each of the four points where the four bodies were joined has been connected with a pin joint, which permits relative rotation of the members. In greater quantity are found applications of linkage where sliding surfaces are introduced.

Here, again, it will be advisable to note the development of the funda-

mental mechanism—the four-bar linkage—into forms where common background is perhaps less readily discernible in the modified design.

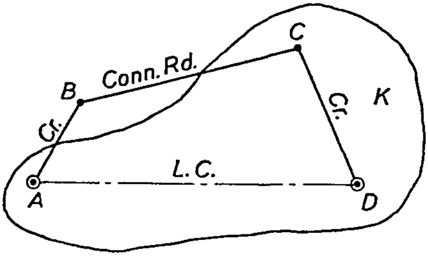


Fig. 260

The fundamental pin-connected form is shown in Fig. 260. The line of centers is a line of body K, the frame of the supporting machine.

The equivalent mechanism shown in Fig. 261 may be substituted with no change in the relative sizes of the four links, and therefore no change in the motion properties.

In this case, a circular slot has been cut in the body K, which is fixed. Point C of the connecting rod is pinned to a small block, S, curved to slide freely in the slot. The center of curvature of the slot is at point D, and its radius is equal to crank CD.

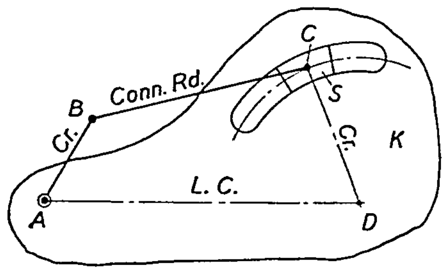


Fig. 261

If crank AB is rotated, the connecting rod will transmit motion to point C. The constraint of the block will force point C to move in a circular path, and radius CD has a motion of pure rotation about fixed axis D. The motion which is thus produced is in every way equivalent to that of a pin-connected four-bar linkage having line of centers AD, a driving crank AB, a connecting rod BC, and a follower crank which is radius CD. The radius CD is a moving line which has no physical existence as an actual body of the mechanism, but serves to analyze the motion, since it behaves, kinematically, in equivalent manner to crank CD of Fig. 260. The background four-bar linkage which underlies the modified physical form has been identified. The fundamental mechanism and applied form are equivalent kinematically, for either will produce the same relative motion of the four members. Since links like crank CD of the altered applica-

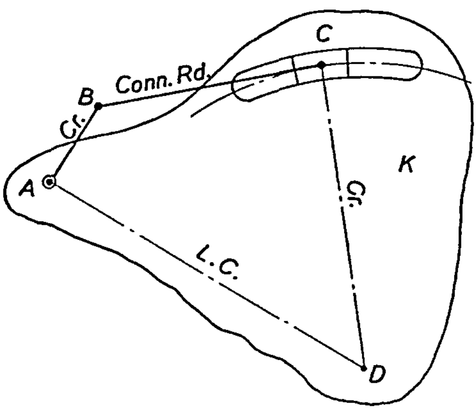


Fig. 262

tion are geometrical lines and not physical or actual bodies, we shall refer to them as *equivalent links*. The term is intended to associate the analysis of linkages with the fundamental linkage which is the common background. Proceeding on this basis, a strong foundation for analysis is available which will serve to give direction to methods of attack as the altered form of mechanisms depart so radically from the pin-connected four-bar linkage that their physical aspect has little resemblance to the original form.

An example of the stages of modification can now serve to trace the transition from pin-connected to sliding forms.

Starting with the linkage of Fig. 261 now identified with its background four-bar linkage, center D is moved in the direction shown in Fig. 262. The line of centers is longer than before. The curved slot, with center at the new point D, still constrains point C to move in a circular path about fixed axis D, and the equivalent crank CD is longer than before. The mechanism has not changed in basic form, but the motion has been affected by the new condition of relative size of the four links. For example, the path of point C, while still curvilinear, is somewhat flatter than it was before. This path will become flatter as point D is moved farther away. At the same time, the line of centers AD and equivalent crank CD are increasing in length, and approaching each other in inclination.

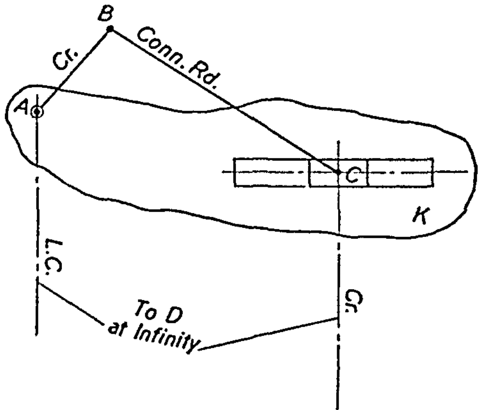


FIG. 263

As D recedes, the radius of curvature of the slot is approaching infinity as a limit, and, in the limit the path of C will become a straight line. Now the linkage is designed as shown in Fig. 263. In establishing its identity with the fundamental four-bar linkage, we note that the equivalent line of centers and equivalent follower crank are both of infinite length, and parallel to each other. In this form, we once again encounter the crank and connecting-rod mechanism.

The relationships of velocity which were developed for a pin-connected four-bar linkage rested upon crank speed ratio. Here we note that the angular velocity of the follower crank is, in the limit, zero, and angular speed ratio has lost its useful significance. The equivalent follower crank has a motion of pure translation, and it is therefore necessary to turn to a study of the linear velocity of the points of this translating line.

In Fig. 264 I_{BC} is the instantaneous axis of connecting rod BC. If the equivalent line of centers is produced to meet the connecting rod produced at point S, then triangles BAS and $BI_{BC}C$ are similar. (The equivalent line of centers and equivalent follower crank are always parallel, and are perpendicular to the sliding surface.)

Then
$$\frac{\text{Velocity of point } C}{\text{Velocity of point } B} = \frac{I_{BC}C}{I_{BC}B} = \frac{AS}{AB}$$

But the velocity of point $B = \omega_{AB} \times AB$

$$\text{velocity of point } C = \omega_{AB} \times AB \times \frac{AS}{AB}$$
$$= \omega_{AB} \times AS.$$

This is the linear velocity of any point which, like C, lies on the equivalent crank. All points of the sliding block, which is moving in pure translation, will have the same velocity.

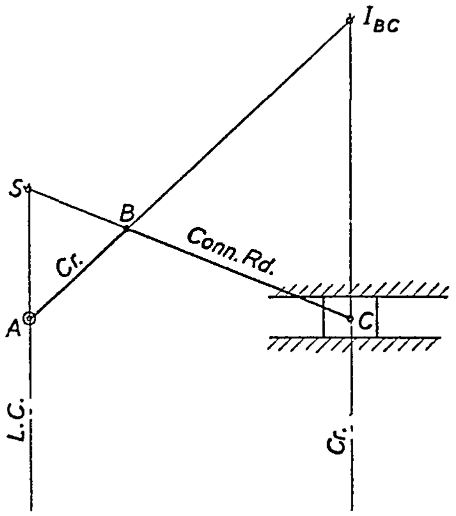


Fig. 264

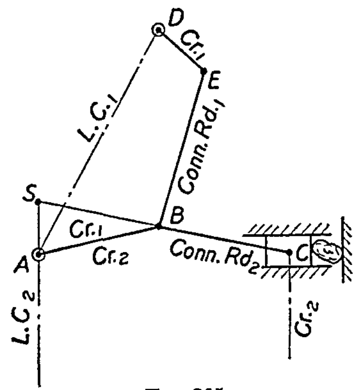


Fig. 265

In the form of crank-and-connecting-rod linkage there exist in practice innumerable applications. The internal combustion engine and the steam engine have already been cited as examples.

The *toggle mechanism* illustrated in Fig. 265 is another example. Here are found two four-bar linkages, one of the pin-connected type whose links are denoted with the subscript *–1* and a crank-and-connecting-rod linkage whose members have been labelled with subscript *–2*.

The toggles are used to take advantage of the fact that the distance AS of Fig. 265 has very low values when the sliding block is near the end of its stroke.

The velocity of C is varying in proportion to the length of AS, and the block will have a very slow motion when C is approaching the end of its stroke. This slow motion results in producing great pressure upon a body placed like the rock of Fig. 265 between a stationary jaw and a moving one forming the sliding block of the linkage.

The *eccentric mechanism* is another modified form of the fundamental four-bar linkage. In Fig. 266 a solid shaft rotates about axis A. If a circular plate, P, is fastened to the shaft, it too must rotate about axis A. An eccentric strap, S, is mounted so that its inner surface may rotate freely relative to the outer surface of plate P, but is carried around with it. The eccentric strap is fixed to a connecting rod, R. Line AB is travelling with pure rotation about fixed axis A, and conforms to the definition of a crank. Line BC is the connecting rod, a line from C perpendicular to the sliding surface moves with the pure translation of an equivalent follower crank, and a line from A parallel to the equivalent follower crank is the equivalent line of centers—this device is the kinematic equivalent of a crank- and connecting-rod linkage.

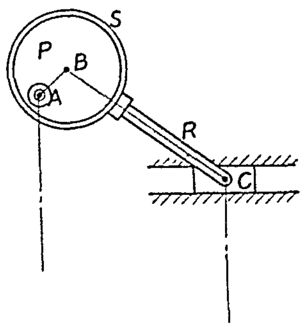
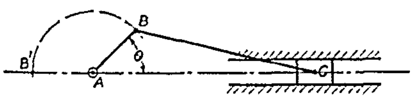


FIG. 266

PROBLEMS

326. Crank AB has an angular velocity of 180 r.p.m., clockwise. Determine the linear velocity of the sliding block when $\theta = 45°$. $AB = 3.0$ in.; $BC = 9.4$ in.

Ans. 245 f.p.m.



PROB. 326

327. Same as Problem 326, with $\theta = 72°$.

328. Same as Problem 326, with $\theta = 120°$.

329. Determine the value of θ in Problem 326, when the sliding block has maximum linear velocity, and the value of the maximum linear velocity.

330. Plot the displacement-time relationship of point C, Problem 326, for one-half revolution of crank AB, starting at position AB'.

Plot the linear velocity-time relationship of point C, determining value by graphical differentiation of the displacement-time curve.

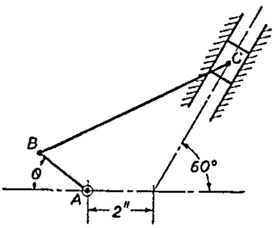
331. Plot the linear velocity-time relationship of point C, Problem 326, for one-half revolution of crank AB, starting at position AB', determining values by the method of Art. 94.

Problems **332–337.** Solve Problems 326–331, with $AB = 2.0$ in.; and $BC = 9.0$ in.

338. Design a crank-and-connecting-rod mechanism with stroke of piston = 4.2 in., ratio of $\dfrac{\text{Connecting rod}}{\text{Crank}} = 3.6 : 1$. Plot a curve showing the velocity-time relationship for the piston.

339. Determine the linear velocity of point C for $\theta = 40°$. Angular velocity of AB = 1 radian per sec. AB = 2.0 in.; BC = 6.5 in.

Ans. 3.3 in. per sec.



Prob. 339

340. Solve Problem 339, for $\theta = 60°$.

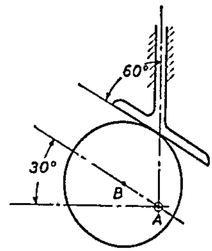
341. Determine the time ratio of $\frac{\text{Forward stroke}}{\text{Return stroke}}$ for point C of Problem 339, if the crank AB rotates at constant angular velocity.

342. Plot the linear velocity-time relationship for point C, Problem 339, for a complete revolution of AB.

343. (a) Identify the equivalent four-bar linkage for the cam-follower drive.

A is the fixed axis of the circular cam, and B is its center. Diameter of cam = 6.0 in.; AB = 2.1 in.

(b) Determine the linear velocity of the follower in the position shown, if the angular velocity of the cam is 1 radian per sec., clockwise.



Prob. 343

95. Modification of Physical Form in Sliding Linkage. Equivalent Kinematic Background. The nature of the transmission of motion from driving crank to follower crank is always governed by control of the relative size of the four members in any four-bar linkage.

The evolution of the crank-and-connecting-rod mechanism from the pin-connected form has identified the four-bar linkage which furnishes kinematical background.

When the slotted piece K of Figs. 261–263 was introduced, sliding contact resulted. With a straight-line sliding surface, two members became infinitely long, while two remained finite. These are shown in Fig. 263. As in the pin-connected form, variation in the nature of the motion transmitted to the follower results from changing the relative size of the links. It is physically possible to fix either of the finite or infinite members, and the fixed member will become the line of centers. For example, infinite member AD may be fixed.

This is effected by pinning axis A to piece K and holding K stationary; and the resulting four-bar linkage is:

$$\begin{aligned} AD &= \text{line of centers} \\ AB &= \text{crank} \\ BC &= \text{connecting rod} \\ CD &= \text{crank.} \end{aligned}$$

The function of K is to provide physically the sliding contact—it is not a member of the equivalent four-bar linkage. If this function of K is appre-

ciated, the identification of kinematic equivalents for mechanisms whose action may at first seem obscure is clarified.

One example of modification of relative size of the four links is illustrated in Fig. 267. Now AB has been fixed. K is pinned to AB at fixed axis A, and

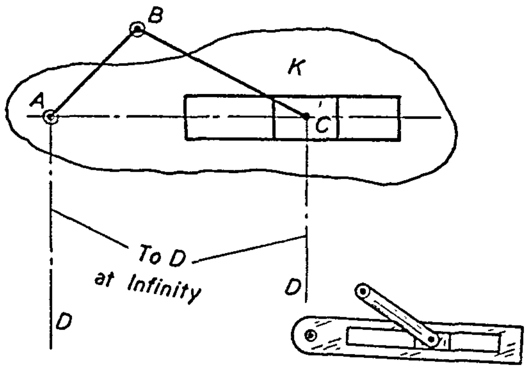


Fig. 267

can have only a motion of pure rotation about A. Body BC is free to rotate about fixed axis B, and therefore must be a crank, conforming to definition. As it rotates, piece K will be forced to rotate about A, and CD, remaining ever perpendicular to the sliding surface, will rotate with K. AD, which must remain parallel to CD, will also rotate—and since its end, A, is a fixed axis, AD must be the other crank. Then CD is a connecting rod, for a connecting rod is defined as the member which joins the moving ends of the two cranks.

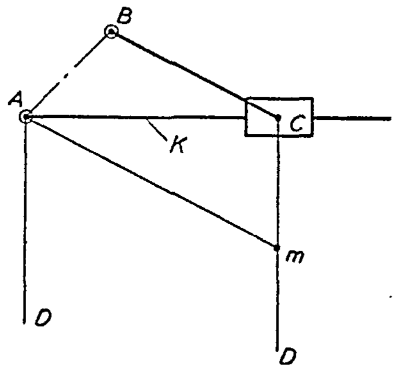


Fig. 268

With the four-bar linkage identified, the study of speed ratio becomes simplified. If, as in Fig. 268, line Am be drawn, for any position of the linkage, parallel to the driving crank BC, then

$$\frac{\omega_{AD}}{\omega_{BC}} = \frac{BC}{Am}.$$

Since piece K and crank AD remain ever perpendicular, they must have the same angular velocity. Then

$$\omega_K = \omega_{BC} \times \frac{BC}{Am}.$$

Another distribution of relative sizes of members may be accomplished by fixing member BC, thereby assigning the function of line of centers

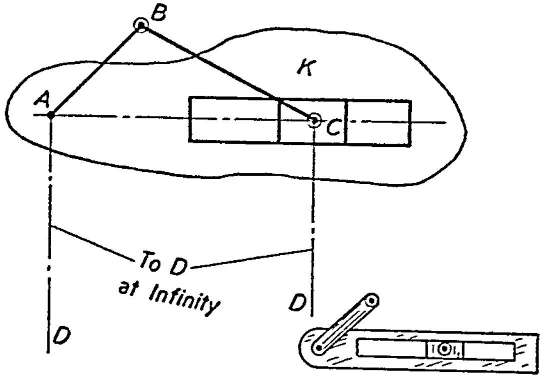


Fig. 269

to it. Now the block at C may have only a motion of pure rotation. If point A of BA is pinned to piece K, the equivalent four-bar linkage appears as in Fig. 269.

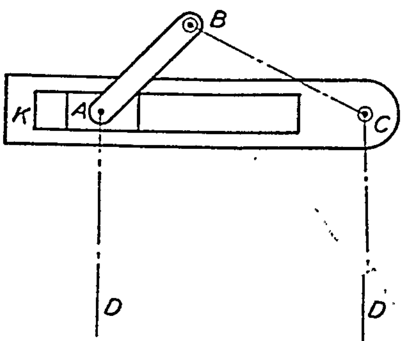


Fig. 270

This form serves as the basis for the Oscillating-Beam quick-return motion discussed in Art. 99. In the application the block is mounted at point A, as shown in Fig. 270. This change does not disturb the identification of the four members, but causes piece K, which is pinned to C, to have a motion of pure rotation.

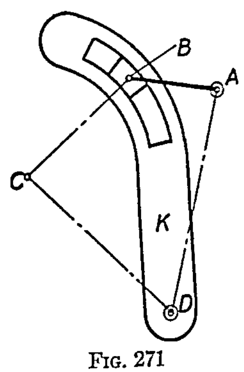


Fig. 271

The previous identifications have been made in cases where the sliding surfaces are straight lines and their radii of curvature infinite, resulting in some infinitely long members of the equivalent four-bar linkage.

With finite radius of sliding surface, similar analysis may be applied, and speed ratios determined after identification of the four members. Piece K of Fig. 271 contains a slot which is circular and has its center of curvature at point C.

Then AB is a crank, since it has a motion of pure rotation about fixed axis A.

Line CD, joining the center of curvature of the sliding surface with fixed axis D, is also a crank.

The connecting rod, defined as the body which joins the moving ends of the cranks, must be BC. AD is the line of centers.

In identifying members, it is important to note that each member must remain of constant length throughout the motion of the linkage, or the relative size distribution is disturbed. In the application just discussed point C remains at constant distance from D and from B, and both the equivalent connecting rod BC and equivalent crank CD satisfy the requirements of proper identification.

The fundamental four-bar linkage may serve, as it has in the examples of mechanism just discussed, to point out a road of attack in analyzing the velocity or acceleration properties. All mechanisms may be shown to rest upon this background, through their property of producing determinate motion. To suggest this basis, however, as a universal tool of analysis would be foolhardy, for altered form occasionally departs so markedly from basic original that the identification of the equivalent four-bar linkage presents great difficulty, and the challenge of motion analysis may be met more directly by other means.

An illustration of this sort occurs in the *Scotch Yoke* linkage illustrated in Fig. 272. The driving crank, AB, is pinned to a block, sliding in a slot of the T-shaped sliding bar, or yoke. The motion of the yoke is one of pure translation. If the crank AB rotates with constant angular velocity, then point E of the yoke will always lie at the foot of a perpendicular drawn from B to the axis of the yoke, which is the direction established by the fixed guides. Then point E and all points on the yoke are travelling in simple harmonic motion (see Art. 101). The solution by the analytical expressions for velocity and acceleration in simple harmonic motion, or by the vector resolution of velocity shown, is simple and direct.

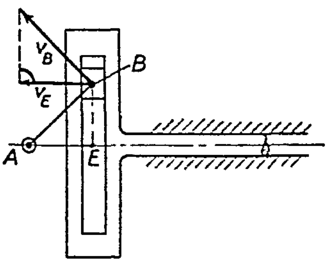


Fig. 272

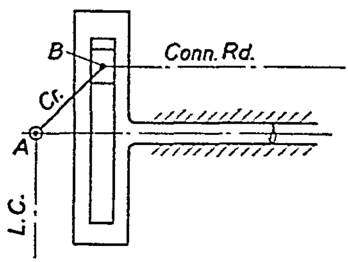


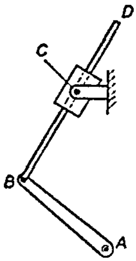
Fig. 273

The identification of the four-bar linkage background is more difficult, and is apt to confuse rather than assist. The results of this identification are shown in Fig. 273. The only link which has physical form is the crank

AB. The equivalent connecting rod is a line from B perpendicular to the sliding surfaces of the yoke guides. The equivalent line of centers joins fixed axis A with another fixed axis at infinite distance. Then the equivalent follower crank is an imaginary line at an infinite distance, parallel to the line of centers.

PROBLEMS

344. Sketch the mechanism in skeleton form, and identify the members of the equivalent four-bar linkage. AB rotates about fixed axis A. The rod BD slides freely through the block, which rotates about a fixed axis at C.

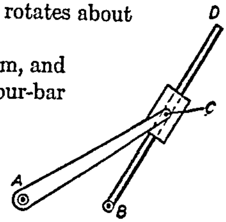


PROB. 344

345. Sketch the mechanism in skeleton form, and identify the members of the equivalent four-bar linkage.

A and B are fixed axes.

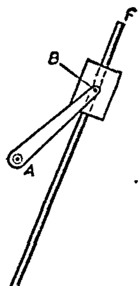
The block at C is pinned to AC, and BD slides freely relative to the block.



PROB. 345

346. Sketch the mechanism in skeleton form, and identify the members of the equivalent four-bar linkage.

A and E are fixed axes. The block at B is pinned to AB, and BF slides freely relative to the block.

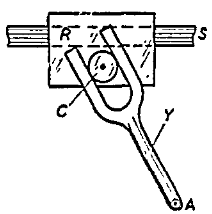


PROB. 346

347. The yoke, Y, rotates about fixed axis A. A roller at C is pinned to the member, R, which slides freely on the fixed guide-bar, S.

Sketch the mechanism, and identify the equivalent four-bar linkage. Assume rolling contact between yoke and roller.

Note: In Problems 344–347, incl., the sketch may be made free-hand. The members of the equivalent linkage are to be identified by lettering the name of each member upon the line drawn to represent it.



PROB. 347

96. Isosceles Sliding Linkage. As in pin-connected four-bar linkages, the modification of the relative sizes of the four basic links is the basis of design for particular forms of motion.

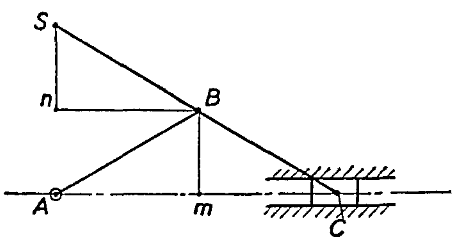


FIG. 274

The *isosceles linkage* involves a special modification of the dimensions of the crank-and-connecting-rod mechanism.

If, as in Fig. 274, the link AB is made equal in length to BC, the resulting mechanism is an isosceles linkage.

If now the connecting rod is produced to point S, making $BS = BC$, the

point S will move in a straight-line path as the driving crank AB rotates. This path will be perpendicular to the line AC.

Let Bn be the X-component of S's displacement from B in any position of the linkage.

Let Am be the X-component of B's displacement from A in the same position.

In right triangles BnS and AmB, nS will always equal mB, and AB is equal to BS. Then $Bn = Am$.

Then S is always remaining directly above A as crank AB turns.

Such a sliding linkage, then, may be used to produce a straight-line path for a tracing point or cutting tool.

There is another property of this isosceles linkage which is useful in generating a special form of path.

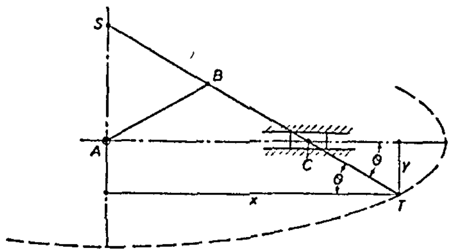


FIG. 275

If the connecting rod is produced to include a point T, as in Fig. 275, the path of point T will be an ellipse, and the linkage may then be used as a drawing instrument for drawing ellipses, or as a cutting tool for cutting elliptical plates.

The equation for the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in which a and b are constants, and x and y the coordinates of the point generating the ellipse from point A taken as an origin.

In this case, let

$$\begin{aligned}
a &= \text{semi-major axis of the ellipse} \\
&= AB + BT = ST \text{ (constant)} \\
b &= \text{semi-minor axis of the ellipse} \\
&= BT - AB = CT \text{ (constant)} \\
x &= ST \cos\theta \\
y &= CT \sin\theta
\end{aligned}$$

$$\frac{x^2}{a^2} = \frac{\overline{ST}^2 \cos^2\theta}{\overline{ST}^2}; \qquad \frac{y^2}{b^2} = \frac{\overline{CT}^2 \sin^2\theta}{\overline{CT}^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2\theta + \sin^2\theta = 1.$$

In any position of the linkage, point T conforms to the equation for points on an ellipse, whose semi-major axis is the sum of AB and BT and whose semi-minor axis the difference between AB and BT.

lines LO and MO forming the cross are sliding relative to the blocks, as they rotate.

Then both of the small blocks are rotating with the same angular velocity.

The Oldham's coupling connects shafts whose axes are at points like S and C. These shafts will therefore have the same angular velocity.

PROBLEMS

348. Design an isosceles linkage to draw an ellipse of 3 in. minor axis, and 5 in. major axis.

349. Design an isosceles linkage to draw an ellipse of 2.8 in. minor axis, and 6.2 in. major axis.

97. Parallel Motion in Sliding Linkages. Pantograph. The linkage illustrated in Fig. 279 is constructed so that members $ABCD$ form a parallelogram. If point E is made a fixed axis, point F, which lies on link AB extended, is free to slide over the background surface. Some point, like S, will then travel in a path similar to that of F. If a line be drawn from F to E, it will intersect CD at point S.

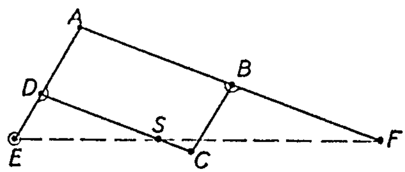


FIG. 279

In the similar triangles ESD and EFA,

$$\frac{SD}{AF} = \frac{ED}{EA}.$$

If F is moved to any new position, as in Fig. 280, a line joining F' and E will intersect $C'D'$ at point S'.

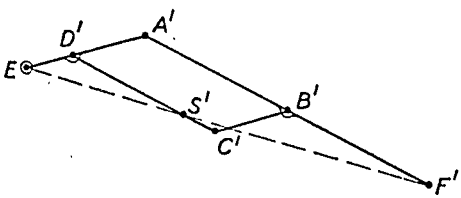


FIG. 280

This point, S', is remaining at a constant distance from $D = SD$. In triangles $ES'D'$ and $EF'A'$ lines $A'B'$ and $C'D'$ remain ever parallel, and the triangles similar.

Then

$$\frac{S'D'}{F'A'} = \frac{ED'}{EA'}.$$

$F'A'$, ED', and EA' are all constant distances, and therefore $S'D'$ is always equal to the original distance SD. Then any straight line like FE, passing through F and the fixed axis E, will always intersect link CD at the same point of CD.

The path of S is therefore paralleling that of F, and their paths are similar. When point S is displaced any distance, point F will travel not only

over a similar path but, in the same time interval, will cover a distance which is related to that traversed by point S in the direct ratio of their respective distances from fixed point E.

The pantograph linkage may then be used as a basis of enlarging (or reducing) drawn figures by placing a marking pencil at F, and a tracing point at S, and fixing point E as an axis. When the tracing point follows the contour of a drawn figure, the marking point at F will draw the similar but enlarged figure. The linkage may also be employed as the basis of a machine tool for cutting large figures from small templates or models serving as a

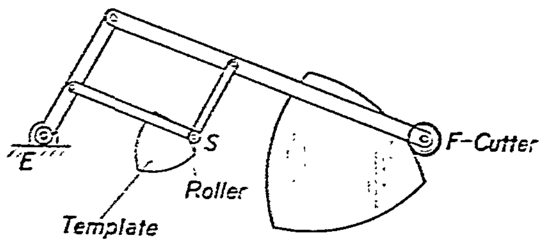
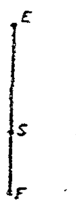


Fig. 281

guide. An example is shown in Fig. 281. A revolving cutting wheel is mounted at F and a small roller at S. The roller is guided around the contour of the template and a similar but enlarged copy is fashioned to shape by the cutting tool.

PROBLEMS



350. Design a pantograph to give a point F (see Fig. 279) a velocity four times as great as that of S.

351. Three points, E, S, and F, are to be joined by a pantograph so that E has a velocity of 5.3 in. per min. to the right, while F has a velocity of 5.3 in. per min. to the left, and S has a velocity of 2.2 in. per min. to the left.

The distance $EF = 8$ in.

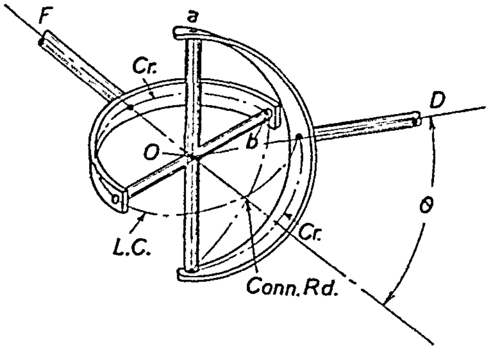
Design the pantograph.

Prob. 351

98. Hooke's Coupling. Universal Joint. Intersecting axes may be connected by a *Hooke's coupling*, illustrated in Fig. 282. The flexibility of the coupling permits transmission between shafts rotating about axes which are fixed in space or between intersecting axes which may move relative to each other. The latter application is found in automotive and machine tool design.

The Hooke's joint is a pin-connected four-bar linkage, and the equivalent linkage has been identified in Fig. 282. It will be noted that the links, unlike the previous examples which dealt only with plane motion, are all portions of great circles of a sphere.

The shafts D and F will make complete revolutions in the same interval of time. During each revolution, however, the speed ratio is not a constant.



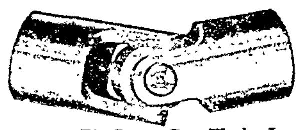

Courtesy The Boston Gear Works, Inc.

FIG. 282

We may derive an expression for this variable speed ratio by taking as in Fig. 283 a plane of projection perpendicular to the axis of the driver. In

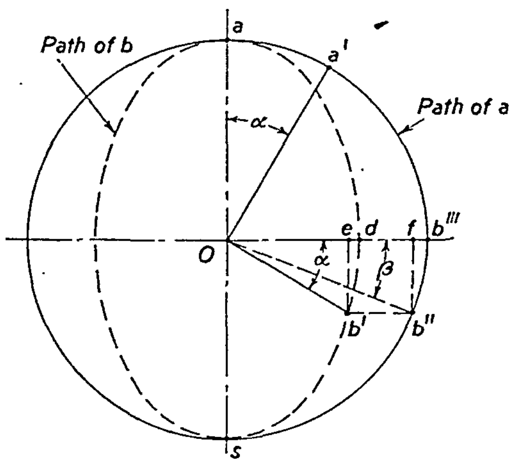


FIG. 283

this plane, the path of point a will be a circle. The projected path of point b on this plane will be an ellipse, since this point has an absolute path which is a circle having a center at O, the common center of the paths of b and a, and an axis which is the axis of F. This ellipse, the projection of a circle of radius Ob, will have a semi-minor axis $Od = Ob \cos \theta$, in which θ is the acute angle between the shafts.

With the two projected paths established, let the driver turn through any angle α, with point a going to position a'. Then point b will move so that its projection is b', because the projections of lines Oa and Ob must remain perpendicular.

The angle $dob' = \alpha$ as shown in the figure is a projection, and the true size of the angle must be established. This may be done by revolving angle dob' about the line as (the major axis of the ellipse) until b' goes to b'', and d moves to b'''. Then the angle β is the true size of the angle, for both of the bounding radii have now been rotated to their true-length positions.

Therefore when the angular displacement of the driver is α, the corresponding angular displacement of the follower is β.

The relationship between α and β may be derived through their tangents.

If we erect a perpendicular $b'e$ to Ob''' from b', and another $b''f$ from b'', we note that

$$\tan \alpha = \frac{b'e}{Oe}$$

and

$$\tan \beta = \frac{b''f}{Of}.$$

Then

$$\frac{\tan \alpha}{\tan \beta} = \frac{b'e}{Oe} \times \frac{Of}{b''f} = \frac{Of}{Oe}.$$

The rotation of angle dOb' to its true-size position carried it through angle θ. During this time projection e moved to f and d to b'''.

Then

$$\frac{Of}{Oe} = \frac{Ob''' = Ob}{Od}$$

But

$$Od = Ob \cos \theta$$

Then

$$\frac{Ob}{Od} = \frac{1}{\cos \theta}$$

$$\frac{\tan \alpha}{\tan \beta} = \frac{1}{\cos \theta}, \qquad \text{and} \qquad \frac{\tan \beta}{\tan \alpha} = \cos \theta.$$

The speed ratio $\frac{\omega_F}{\omega_D}$ may now be obtained by differentiation, noting that $\cos \theta$ remains a constant.

$$\frac{\sec^2 \beta d\beta}{\sec^2 \alpha d\alpha} = \cos \theta$$

$$\frac{d\beta}{d\alpha} = \frac{\cos \theta \sec^2 \alpha}{\sec^2 \beta} = \frac{\cos \theta \sec^2 \alpha}{1 + \tan^2 \beta}$$

β may be eliminated.

Since $$\tan \beta = \tan \alpha \cos \theta$$

$$\frac{\omega_F}{\omega_D} = \frac{d\beta}{d\alpha}$$

$$= \frac{\cos \theta \sec^2 \alpha}{1 + \tan^2 \alpha \cos^2 \theta} = \frac{\cos \theta}{\cos^2 \alpha + \sin^2 \alpha \cos^2 \theta}$$

$$= \frac{\cos \theta}{\cos^2 \alpha + \sin^2 \alpha \,(1 - \sin^2 \theta)}$$

$$= \frac{\cos \theta}{1 - \sin^2 \alpha \sin^2 \theta}.$$

This expression may be used to find the speed ratio when the driver has turned through any angle α.

Since the follower of this Hooke's coupling has a variable speed, it forms an objectionable means of transmission when constant speed ratio must be preserved at all instantaneous positions.

The double *Universal joint* is formed by transmitting the motion from a driver to follower through an intermediate shaft. The variation of velocity of the single Hooke's coupling which joins the driver and intermediate shaft is exactly compensated by a second Hooke's coupling placed between intermediate shaft and follower shaft. The intermediate shaft must be placed so as to make the same angle with driver and follower shafts and the forks at the ends of the intermediate shaft must lie in the same plane.

PROBLEMS

352. Two shafts, A and B, are connected by a Hooke's joint. The acute angle between the shafts is 40°. Shaft A has constant angular velocity.

Plot a curve showing the angular displacements of A as abscissae, and the angular displacements of B as ordinates.

Use 15° intervals of A's displacement during a complete revolution for determining values.

353. If the shaft A of Problem 352 has a speed of 100 r.p.m., plot the angular velocity-time relationship of shaft B for a complete revolution.

354. Two shafts are connected by a Hooke's joint. The driving shaft has a constant angular velocity of 120 r.p.m. What is the maximum permissible angle between the shafts if the maximum speed of the follower shaft is to be 125 r.p.m.?

355. A drive consists of two shafts connected by a Hooke's joint. The drive shaft has a constant angular velocity, and the follower shaft is to have an angular velocity which must not vary more than 5 per cent above or below its mean speed. Determine the maximum permissible angle between the shafts.

99. Series of Four-Bar Linkages. Early in the development of linkages as practical expressions of determinate motion, the possibilities of series or

multiple arrangements of linkage combinations became evident. They permit great flexibility in the nature of the variation of speed ratio, or may be used to overcome difficulties arising from the use of a single four-bar linkage.

For example, when a crank-and-connecting-rod mechanism is used in a steam engine, the block cannot be used as a driver to produce a constant circular motion of the crank unless some means is provided of passing the "dead points" at the ends of the stroke. A reservoir of energy, like a flywheel, may be employed to effect continuous motion by the dead points, when circular motion of the crank has once been established. The problem remains of starting the motion, in the case of such an engine which has come to rest at the dead point.

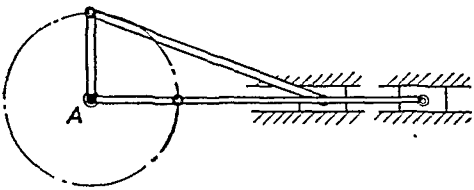


FIG. 284

This difficulty may be removed by using a parallel arrangement of two such four-bar linkages, as in Fig. 284 where the two cranks fastened to the same crankshaft A are placed at right angles to each other. When one sliding block is at the end of its stroke, the other is in such position that the introduction of pressure upon it starts the crankshaft. Such arrangements are found in locomotive and marine engines.

A most extensive practical application of multiple arrangements is made in the case of quick-return linkages. In the design of reciprocating tools the advantages presented by the provision of a rapid return stroke have already been discussed.

One application of series of linkages is made in the *Oscillating-Beam* quick-return mechanism of Fig. 285. The first linkage with its four members has been identified in Fig. 286 as linkage *1*. The *driving crank AB rotates at constant angular velocity*, causing beam ST to oscillate. The ratio of time of cutting stroke to time of return stroke equals the ratio of angle α to angle β. These angular displacements are fixed in magnitude by the points L and M where the center line of the oscillating beam is tangent to the circular path of point B.

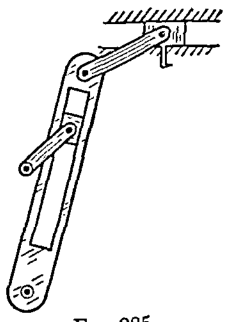


FIG. 285

A second four-bar linkage, whose members are labelled *2* in the figure, transmits the motion from the oscillating beam to the cutting tool E. In such uses as machine tools, where the length of desired cutting stroke may vary, control is readily accomplished by increasing or reducing the length of the driving crank.

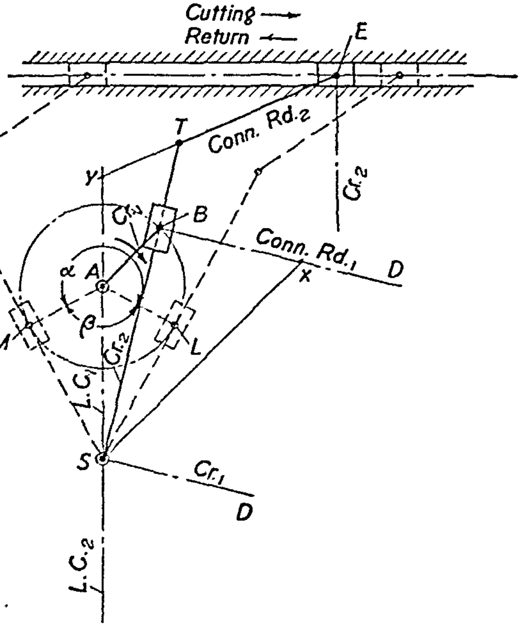


FIG. 286

This linkage affords an excellent opportunity of analyzing the velocity relationships in a series arrangement of linkages by identifying the equivalent four-bar linkages. In the case of linkage *1* the speed ratio for the cranks may be determined by the method discussed in Art. 88.

The angular speed of the oscillating beam is ω_{ST}, and is equal to the angular speed of the equivalent follower crank, ω_{SD}, for this follower crank must always be perpendicular to the surface of sliding.

Then if the angular speed of the driving crank is ω_{AB},

$$\frac{\omega_{ST}}{\omega_{AB}} = \frac{AB}{Sx} \text{ (}Sx\text{ parallel to }AB\text{)}$$

or

$$\omega_{ST} = \omega_{AB} \times \frac{AB}{Sx}.$$

In the second four-bar linkage, the crank-and-connecting-rod appears as equivalent linkage. The linear speed of point T is

$$\omega_{ST} \times ST.$$

Then

$$\frac{\text{Linear speed of } E}{\text{Linear speed of } T} = \frac{Sy}{ST}$$

and $$\text{Linear speed of } E = \text{Linear speed of } T \times \frac{Sy}{ST}$$

$$= \omega_{ST} \times ST \times \frac{Sy}{ST} = \omega_{AB} \times \frac{AB}{Sx} \times Sy.$$

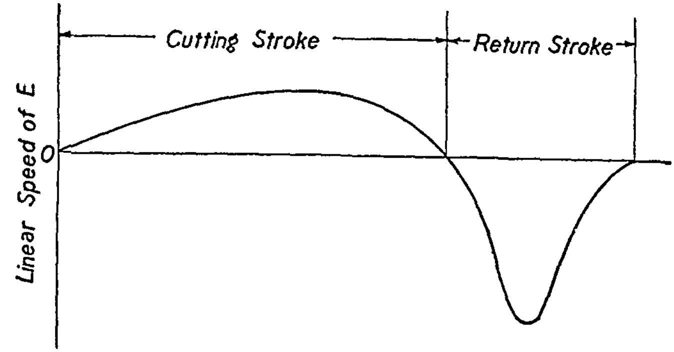


Fig. 287

This type of equation expresses the linear speed of E as a function of the angular speed of AB and a series of distances which may be determined by drawing the mechanism in a series of positions. Such an expression affords a much simpler method of analysis than the resolution of velocity vectors, when the mechanism is to be studied in many positions.

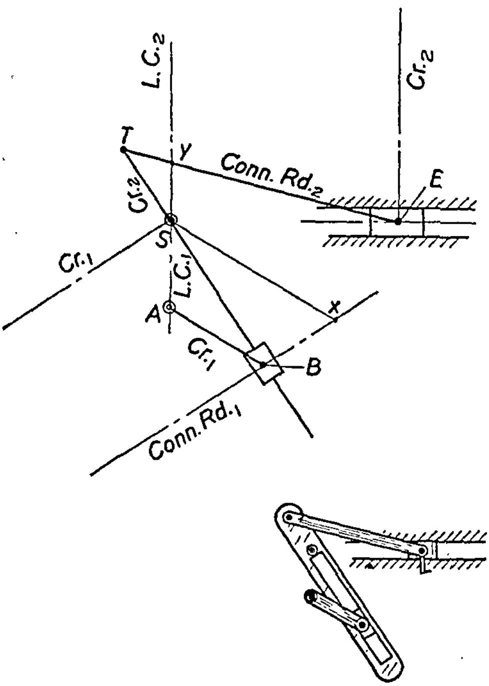


Fig. 288

ω_{AB} and AB are of constant value. It is then necessary to measure only two lines in any position of the mechanism. The equivalent connecting rod of linkage *1* must be produced to enable Sx to be measured, and the connecting rod of linkage *2* must be produced to yield Sy. When these two distances are measured, they may be multiplied by the constant product of ω_{AB} and AB. The curve of Fig. 287 shows the nature of the linear speed of E for a constant angular velocity of the driv-

ing crank AB and a ratio of cutting stroke to return stroke = 2 to 1.

The *Whitworth Quick Return* makes use of a rotating beam, instead of an oscillating one as in the Oscillating Beam Quick Return. This series of linkages is illustrated in Fig. 288 which shows both the form of the practical application, and the skeletonized form, together with the identification of the two equivalent linkages. The expression for linear speed of E, which is the ram carrying a cutting tool, is derived, after identification of equivalent linkages, in the same manner as that employed in the case of the Oscillating-Beam Mechanism.

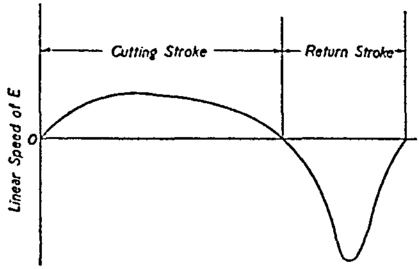
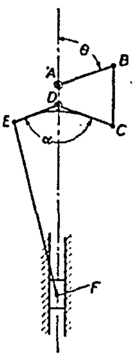


FIG. 289

Fig. 289 shows the nature of the linear speed of the ram for constant angular velocity of the driving crank.

PROBLEMS

356. The skeleton mechanism shown is taken from a stamping machine. The driving crank AB has a constant angular velocity of 240 r.p.m., counter-clockwise. $AB = BC = CD = 3.0$ in.; $AD = 1.2$ in. A and D are fixed axes. $DE = 2.4$ in.; $EF = 12.0$ in. F moves in fixed guides, CDE is a rocker arm.



PROB. 356

(a) Plot the angular displacement-time curve for crank CD, starting in the position when $\theta = 0°$.

(b) From the plotted curve determine the angle α at which the rocker arm should be set to give a maximum value of the time ratio of $\dfrac{\text{Forward stroke}}{\text{Return stroke}}$ of F.

357. If the rocker arm CDE of Problem 356 is set at $\alpha = 135°$, determine the time ratio of $\dfrac{\text{Forward stroke}}{\text{Return stroke}}$ of F.

Plot a curve of the displacement-time relationship of F for constant angular velocity of $AB = 1$ radian per sec., clockwise.

Plot the velocity-time curve for F, obtaining values by graphical differentiation of the displacement-time curve.

358. Using the data of Problem 357, identify the members of the equivalent four-bar linkages, and derive an expression for the linear speed of F as a function of the angular velocity of AB. The equation is to be of the same type as that derived in Art. 99 for the mechanism of Fig. 285.

Draw the mechanism in a series of positions and draw and measure the variable factors of the derived equation for the linear speed of F.

Plot the velocity-time curve for F.

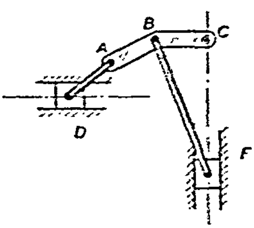
A sufficient number of positions should be drawn to yield a smooth curve.

359. The drive mechanism of a pneumatic hammer is shown. The driving piston, D, slides in fixed horizontal guides, and the ram, F, slides in fixed vertical guides. Axis C is fixed. $AB = BC = 3.2$ in.; $AD = 3.6$ in.; $BF = 10.0$ in.; Angle $CBA = 150°$.

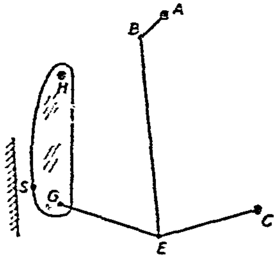
The center line of the slot at D is 3.6 in. below C.

Identify the equivalent four-bar linkage and derive an expression for the linear speed of F as a function of the linear speed of D.

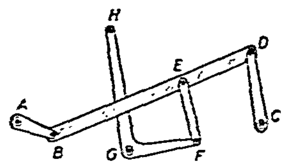
Determine the linear speed of F in the position shown (BC horizontal) if D has a linear speed of 10 in. per sec. What is the mechanical advantage for this position?



PROB. 359



PROB. 360



PROB. 361

360. The stone-crusher mechanism is driven by crank AB.

Fixed axis C is 15 in. to the right of A and 30 in. below it.

Fixed axis H is 16 in. to the left of A, and 10 in. below it.

$AB = 4$ in.; $BE = 31$ in.; $CE = 16$ in.; $GE = 16$ in.; $GH = 20$ in.

Point S on the jaw of the crusher is 5 in. from G, and 18 in. from H.

Identify the equivalent four-bar linkages and derive an expression for the linear speed of S as a function of the angular speed of AB.

If crank AB rotates with constant angular velocity of 90 r.p.m., clockwise, plot the speed-time curve of point S for a complete revolution of AB.

361. A feed mechanism for a candy-wrapping machine is driven by crank AB. The motion of point H is used to feed the paper into the wrapping machine.

A, C, and G are fixed axes.

$AC = 8$ in.; $AG = 3.7$ in.; $CG = 4.5$ in.

$AB = 1.2$ in.; $BD = 7.0$ in.; $CD = 2.3$ in.; $DE = 2.5$ in.; $EF = 1.9$ in.; $FG = 2.3$ in.; $GH = 3.8$ in. The angle FGH of the rocker arm is 90°.

Identify the members of the equivalent four-bar linkages, and derive an expression for the angular velocity of GH as a function of the angular velocity of crank AB.

Plot the angular velocity-time relationship of GH for a complete revolution of AB, which has an angular velocity of 120 r.p.m., clockwise.

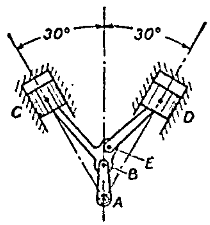
362. The crank-and-connecting-rod arrangement of a pair of cylinders in a V-type engine is shown in the diagram. $AB = 4$ in.; $BE = 2.0$ in.; $ED = 7.3$ in.; $BC = 9.3$ in.; Angle $EBC = 60°$.

(a) Identify the background series of equivalent four-bar linkages.

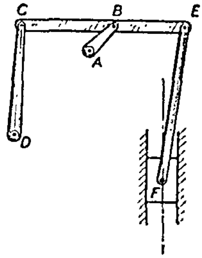
(b) Derive an expression for the linear speed of C as a function of the angular speed of AB, and determine from it the speed of C in the position shown with AB rotating clockwise at 1800 r.p.m.

(c) Derive an expression for the linear speed of D as a function of the angular speed of AB, and determine from it the speed of D in the position shown with AB rotating clockwise at 1800 r.p.m.

(d) Derive an expression for the linear speed of D as a function of the linear speed of C, and check the results of parts (b) and (c) from the expression.



Prob. 362



Prob. 363

363. The linkage shown is used for obtaining a stroke of a piston, F, which is much longer than the crank radius AB.

CD rotates about fixed axis D, and AB about fixed axis A. Piston F slides in fixed guides.

The vertical center line of F is 4.8 in. to the right of point A. D is 4.9 in. to the left of and 6.0 in. below A.

AB = 2.3 in.; BC = 6.0 in.; CD = 7.2 in.; BE = 4.6 in.; EF = 10.0 in.

(a) Determine the length of stroke of F.

(b) Identify the background series of equivalent linkages.

(c) Derive an expression for the linear speed of F as a function of the angular speed of AB.

(d) Determine the linear speed of F if AB has an angular velocity of 60 r.p.m., clockwise, when AB is in each of its horizontal positions.

364. Determine the crank length, AB, of the linkage shown in Problem 363 to give F a stroke of 5 in. All other dimensions remain unchanged.

365. Design an Oscillating-Beam Quick Return Mechanism (see Fig. 285) to fulfill the following conditions:

(a) Stroke of the ram E = 12 in.

(b) Time ratio of $\dfrac{\text{Cutting stroke}}{\text{Return stroke}} = 2.0$ to 1.

Plot a curve giving the speed-time relationship of the tool for unit angular velocity of the driving crank for one complete revolution.

366. If the driving crank of the quick-return mechanism of Problem 365 is adjustable, determine the time ratio of $\dfrac{\text{Cutting stroke}}{\text{Return stroke}}$ if:

(a) Stroke = 6.0 in.

(b) Stroke = 8.4 in.

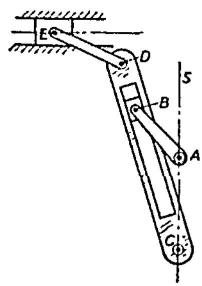
(c) Stroke = 10.0 in.

In each case, report the length of the crank setting.

367. Time ratio of $\dfrac{\text{Cutting stroke}}{\text{Return stroke}} = 2.6$ to 1. Center line of slide E is 6.5 in. above A.

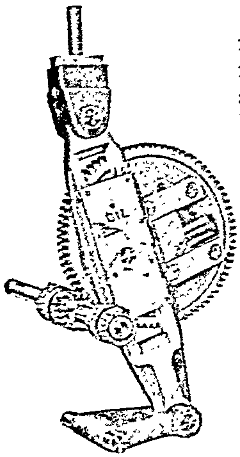
AB = 5.2 in.; CD = 16 in.; DE = 6 in. Locate point C.

If AB has angular velocity of 60 r.p.m., clockwise, plot a curve giving the speed-time relationship for E, determining values by identifying the equivalent four-bar linkages, and deriving the equation of relationship between the linear speed of E and the angular velocity of AB.



Prob. 367

368. A re-design of the mechanism of Problem 367 is to be made by a tool manufacturing concern. The objective of the new design is to give a more uniform speed to the cutting stroke. It will be noted from the derived expression of Problem 367 that if crank length AB is made variable, it may be adjusted to compensate for the variable factors of the expression. Design a pitch surface for a cam to be used as driving crank, replacing AB to afford an approximately uniform speed of cutting stroke.



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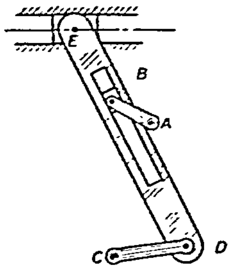


Prob. 369

369. A ram-driving mechanism taken from a shaper is shown.

A and C are fixed axes, and E slides in horizontal fixed guides.

The center line of the slider at E is 8.5 in. above A.

C is 12.5 in. below, and 2.4 in. to the left of A.

$AB = 4.2$ in.; $CD = 7.0$ in.; $DE = 23.0$ in.

Identify the equivalent four-bar linkages, and determine the time ratio of $\dfrac{\text{Cutting stroke}}{\text{Return stroke}}$ if AB rotates with constant angular velocity.

370. A Whitworth Quick-Return Mechanism used in a textile machine has a driving crank AB, which rotates at 150 r.p.m., clockwise.

A and D are fixed axes. The piece CDE rotates about D, and slides freely in a slot at B.

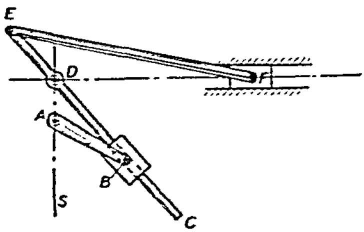
Point F slides in fixed horizontal guides.

CDE makes complete revolutions about D.

$AB = 5.2$ in.; $DE = 8.0$ in.; $EF = 20.0$ in.

Locate point D, if the time ratio of the $\dfrac{\text{Forward stroke}}{\text{Return stroke}}$ is 2 to 1.

Plot the curve of speed-time relationship for point F.



Prob. 370

371. Redesign the linkage of Problem 370 for a time ratio of $\dfrac{\text{Forward stroke}}{\text{Return stroke}} = 2.4$ to 1.

372. Redesign the linkage of Problem 370 for a time ratio of $\dfrac{\text{Forward stroke}}{\text{Return stroke}} = 1.7$ to 1.

373. Determine the maximum speed of F of Problem 372.

VII

ACCELERATION

100. Acceleration. The preceding investigations of kinematic properties have been concerned with change of position—displacement—and the time rate of change of position—velocity. Velocity may itself be variable, and changes in velocity with respect to time may now be explored.

Acceleration is defined as the *time rate of change of velocity.*

Velocity is a vector quantity. Then changes in velocity may be changes in magnitude, in direction, or in both magnitude and direction. These changes must be described by noting the degree of change of both a magnitude and a direction. Acceleration is therefore a vector quantity.

101. Linear Acceleration. The simplest form in which the kinematic property of acceleration is encountered arises in the case of a particle which has rectilinear motion.

Such a particle will have but one degree of freedom; its inclination is therefore fixed, and its velocity may vary in magnitude or in sense only.

When a change of velocity, Δv, takes place in time Δt, the average acceleration or rate of change with respect to time then must be $a = \frac{\Delta v}{\Delta t}$ and, as indefinitely smaller intervals of time are considered,

$$a = \operatorname*{Limit}_{\Delta t \doteq 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

or, acceleration is the first derivative of velocity with respect to time.

It has already been noted that

$$v = \frac{ds}{dt}.$$

Then
$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

or, acceleration is the second derivative of displacement with respect to time.

These expressions furnish a basis for relating a, v, s, and t. It should be noted that the process of differentiation employed yields only the *magnitude* of the acceleration.

A modified form of the relationship is also useful.

Since

$$v = \frac{ds}{dt}$$

and

$$a = \frac{dv}{dt}$$

$$dt = \frac{ds}{v} = \frac{dv}{a}$$

and

$$vdv = ads.$$

These basic relationships become simplified when the acceleration is *constant.* If in this case the velocity at the beginning of a time interval is called v_o, and the velocity at the end of the time interval v_f, the acceleration,

$$a = \frac{v_f - v_o}{t}$$

or

$$v_f = v_o + at.$$

The distance, s, travelled during the time interval, t, with acceleration constant, will be the product of the average velocity during the time interval and the time, or

$$s = \frac{v_o + v_f}{2} t.$$

Since the velocity is changing uniformly,

$$s = \frac{v_o + v_f}{2} t = \frac{v_o + (v_o + at)}{2} t$$

Then

$$s = v_o t + \tfrac{1}{2} at^2.$$

If, as in the case of a body falling freely under the influence of gravity, which is an example of constant acceleration, the displacements for successive equal time intervals are investigated, they will be found to be related in geometric progression. Such a body, starting from rest, will have displacement $s = \frac{1}{2}at^2$. Then, in the first interval of time, it will have been displaced one unit of distance. In the next equal time interval, it will have been displaced three more of these same units of distance; during the third equal time interval, five more. The progression will be in the continuing order 1 : 3 : 5 : 7 : 9 etc.

In mechanism applications, as in the design of a cam's displacement schedule, this form of accelerated motion is frequently used, and it will be convenient to have available a graphical method of determining displacements.

A particle starts from rest at point A, Fig. 290, and travels to point B in 5 seconds with constant acceleration. The displacements are in the ratio of 1 : 3 : 5 : 7 : 9.

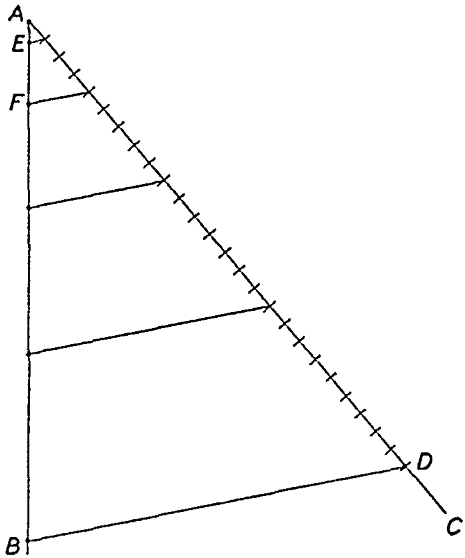


Fig. 290

To locate, graphically, the position of the particle at the end of each second, we use similar triangles in setting the proportion. A line AC is drawn, making any angle with AB, and equal units of distance are laid off along AC. Any convenient unit may be used. From the end, D, of the 25th unit a line DB is drawn to the final position, B, of the moving particle. Now parallels to DB are drawn from the ends of the 1st, 4th, 9th, and 16th units. These parallels divide line AB into segments whose lengths are in the desired ratio 1 : 3 : 5 : 7 : 9. At the end of the first second the particle will have arrived at point E, at the end of the second second it will be at point F, and so on.

Another relationship of constant acceleration may be derived by eliminating the time element, t.

$$v_f = v_o + at$$

and

$$v_f^2 = v_o^2 + 2\,v_o at + a^2t^2$$

$$s = v_o t + \tfrac{1}{2}\,at^2$$

and

$$2\,as = 2\,av_o t + a^2t^2.$$

Then

$$v_f^2 = v_o^2 + 2\,as.$$

When the linear acceleration is *variable*, other expressions than those given above must of course be used.

In general the calculus must be employed. If the equation for velocity-time is known, we differentiate and have an expression for acceleration. Or, if the known material presents an equation of relationship between displacement and time, a first differentiation yields the velocity-time relationship, and the second differentiation establishes the acceleration.

Many mechanisms, particularly of the linkage type, produce a linear motion in which the equation expressing relationship between displacement and time cannot readily be obtained. In such cases, there is available the graphical equivalent for analytical differentiation. This method was discussed in the chapter on velocities (see Art. 42) and may be employed to obtain acceleration.

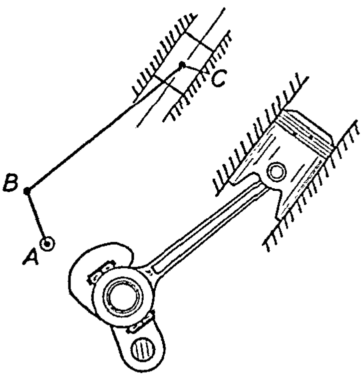


Fig. 291

In the crank-and-connecting-rod mechanism of Fig. 291 the dimensions of the linkage and the angular velocity of the driving crank AB are known. Then by drawing the mechanism in several positions a curve may be established which gives the relationship between the displacement of point C and time.

Setting tangents to this curve will produce the velocity-time relationship, which is shown in Fig. 292. The velocity-time curve may also be obtained by the method of Art. 94. If this latter curve is again differentiated by the

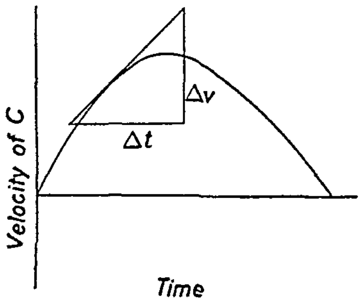


Fig. 292

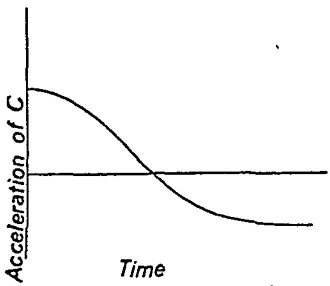


Fig. 293

tangent method, the acceleration-time curve may be plotted. Figure 293 shows the resulting acceleration-time curve. This method of establishing the acceleration is a basic and simple one. It is capable of universal application, and is the most adequate method for general investigations of acceleration in mechanism applications.

Simple Harmonic Motion. A special form of variable acceleration is encountered frequently, both in linkage applications and as a basic factor in vibratory motions. This is simple harmonic motion. In this case a particle moves along a straight line so that its acceleration is always proportional to its displacement, x, from any fixed point in the line which may be used as a reference point; and the sense of the acceleration is always towards the reference point.

Expressed mathematically,

$$a = \frac{d^2x}{dt^2} = -Kx$$

where K is a constant and the minus sign indicates that the sense of the acceleration is always opposite to that of the displacement x. For example,

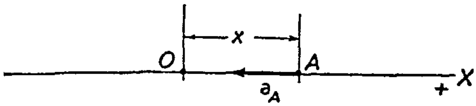


FIG. 294

in Fig. 294, point O has been selected as an origin. Then the point A is displaced a distance x from O which is a positive displacement by the conventions set along the axis. The acceleration will be in the negative direction, or its sense will always be directed towards point O.

When a particle B is moving along a circular path, at the end of a radius, r, which has constant angular velocity, ω, as in Fig. 295, then the motion of

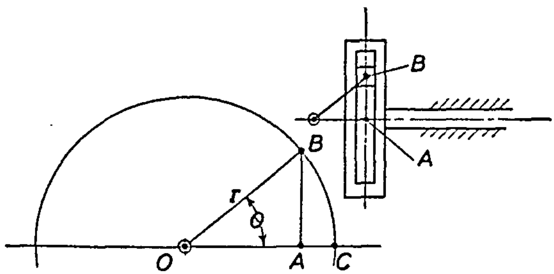


FIG. 295

the point A, which is the projection of point B on the diameter of the circle, is simple harmonic motion.

If t is the time interval during which the point A moves from position C to position A, then $\theta = \omega t$

and $$x = r \cos \theta = r \cos \omega t$$

The velocity of point A

$$v_A = \frac{dx}{dt} = -\omega r \sin \omega t$$

and A's acceleration,

$$a_A = \frac{d^2x}{dt^2} = -\omega^2 r \cos \omega t = -\omega^2 x.$$

Then the motion of A is simple harmonic motion, and the constant K of the basic expression $a = -Kx$ is ω^2, which is the square of the constant angular velocity of the rotating radius.

The graphical analysis of simple harmonic motion is illustrated in Fig. 296. A particle is travelling along line AB with simple harmonic motion, and goes from A to B in six seconds. It is desired that the position of the moving particle at the end of each second be located.

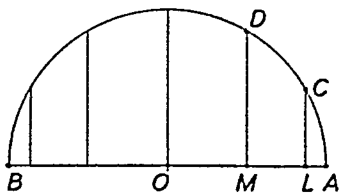


Fig. 296

With O, the mid-point of AB, as center and AB as diameter, a semicircle is drawn. The semi-circumference is next divided into six equal arcs, as AC, CD, etc. From each point C, D, etc., a perpendicular is dropped to intersect AB at points L, M, etc. Then at the end of the first second after leaving A, the particle will pass point L; at the end of the second second, it will pass point M; etc.

PROBLEMS

374. A particle travels along a straight path, according to the following equation of motion:

$$s = 6\,t^2 - 2\,t^3$$

where s is the displacement in inches, and t is time in seconds.

Plot the displacement-time curve for the first 3 sec. starting at $t = 0$.

Plot the acceleration-time curve, determining values by graphical differentiation, for the first 3 sec. starting at $t = 0$.

375. Solve Problem 374, changing the equation of motion to:

$$s = 3\,t^3 - 5\,t^2 + 3\,t.$$

376. A particle travels along a path which is 12 in. long. During the first sec. it goes 4 in. with constant acceleration; during the second sec. it goes 3.5 in. with constant velocity; during the third sec. it goes the remaining 4.5 in., decelerating with simple harmonic motion, coming to rest at the end of the path. It has zero initial velocity.

Determine the displacement of the particle at the end of each interval of 0.25 secs., graphically.

377. A particle travels along a straight vertical path starting from rest at one end of the path. The total length of path = 6.0 in., and the total time to travel to the other end of the path and return to the starting position is 6 sec.

During the first sec., the particle has constant acceleration, and goes 4 in.; during the second sec. it travels 2 in. with constant deceleration to the end of the path. It next dwells for 1 sec., and then returns to the starting position in 3 sec. with simple harmonic motion.

Determine graphically the displacement of the particle from its starting position at time $t = 0$ for each $\frac{1}{4}$ sec. interval.

378. Design a plate cam which will give the described motion to the particle of Problem 377. Use pointed follower, with cam axis in line with path of follower. Starting position of tip of follower 1.0 in. above cam axis.

379. Solve Problem 378, using a roller follower, with cam axis 2.0 in. below and 2.4 in. to the right of the starting position of the center of the follower. The description of particle displacement given in Problem 378 applies to the center of the roller. Roller diameter = 1.0 in.

380. A particle moves in a straight line with the velocity-time curve whose values are given in Problem 107.

Plot the velocity-time curve, and differentiate, graphically, to obtain values for an acceleration-time curve. Plot the acceleration-time curve.

381. Solve Problem 380 using the values of velocity and time given in Problem 108.

382. Solve Problem 380 using the values of velocity and time given in Problem 109.

383. Plot the acceleration-time curve of point C on the sliding block of a crank-and-connecting-rod mechanism, for one stroke.

The driving crank, AB, rotates at constant angular velocity of 1 radian per sec., clockwise. $AB = 2$ in.; $BC = 8$ in.

(a) Determine values by graphical differentiation.

(b) Determine the position of crank AB for maximum velocity of point C.

384. Plot the acceleration-time curve of the cam follower of Problem 103. Determine values by graphical differentiation.

385. Design a plate cam, with pointed follower, if the follower is to have the following motion: the path of the follower is a straight vertical line. Follower rises 6 in. with simple harmonic motion in 3 sec.; dwells for 1.5 sec., drops 2 in. at once; then returns to the starting position in 2.5 sec., with constant acceleration for 1.5 sec., and constant deceleration for 1.0 sec.

The cam rotates clockwise, making one revolution in 7.0 sec., and the cam axis is 2.0 in. directly below the starting position of the follower.

386. Solve Problem 385 if the follower is a roller follower. Cam axis 3.2 in. directly below the starting position of the center of the roller follower. Roller diameter = ¾ in. Cam rotates counter-clockwise.

387. Plot the displacement-time curve for point E of Problem 367. Plot the acceleration-time curve, determining values by graphical differentiation.

388. Solve Problem 387, using the mechanism and data of Problem 369.

389. Plot the displacement-time curve for point F of Problem 370. Plot the acceleration-time curve, determining values by graphical differentiation.

390. The following table gives the velocity-time relationship of a particle which moves in a straight line.

Determine the acceleration-time curve.

Velocity	*Time*
0 in. per sec.	0 sec.
1	1.91
2	3.58
3	4.92
4	5.76
5	6.07 (max.)
6	5.89
7	5.30
8	4.47
9	3.44
10	2.31
11	1.16
12	0

391. The following table gives the values of the acceleration of a piston against crank angles. The crank has a constant angular speed of 240 r.p.m.

Plot the acceleration-time curve and determine the stroke of the piston by graphical integration.

Acceleration	*Crank angle*
329 ft. per sec. per sec.	0 deg.
305	15
262	30
190	45
99	60
8	75
−68	90
−124	105
−165	120
−185	135
−194	150
−196	165
−197	180

102. Acceleration in Curvilinear Motion. When a particle moves with curvilinear motion, the velocity is changing in direction, and its magnitude or speed may be constant or changing.

In the example shown in Fig. 297, a particle travels in a circular path about O, with constant speed. The direction of its velocity will be constantly changing.

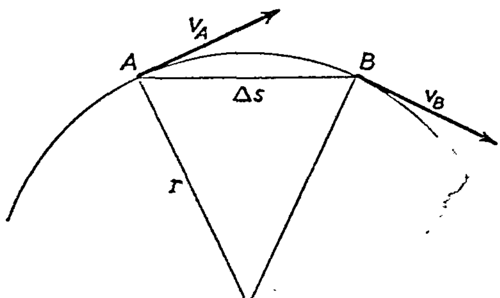


Fig. 297

At A, the velocity of the particle is v_A, and at B it is v_B. The difference between these two velocities is $\Delta v = v_B - v_A$, the vector difference obtained

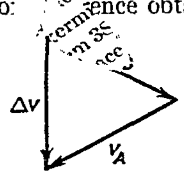


Fig. 298

in Fig. 298. The vector-triangle of Fig. 298 is similar to triangle OAB of Fig. 297, because corresponding angles are equal.

Then
$$\frac{\Delta v}{v_A} = \frac{\Delta s}{r}.$$

As angle AOB is indefinitely decreased, Δv approaches as its limit dv, and Δs approaches ds. In the limit:

$$\frac{dv}{v} = \frac{ds}{r}$$

But
$$ds = vdt$$

and
$$\frac{dv}{v} = \frac{vdt}{r}$$

or
$$\frac{dv}{dt} = \frac{v^2}{r}.$$

That is,
$$a = \frac{v^2}{r}.$$

This acceleration is in the direction of radius r. Note that the chord Δs will approach a perpendicular to radius OA as its limiting direction, and Δv in the similar triangle will become perpendicular to v_A at its limit.

This acceleration in a radial direction is called *normal acceleration*, and will be symbolized as a_n.

When the particle moves in a circular path and has a normal acceleration, $a_n = \frac{v^2}{r}$, but is no longer moving with constant speed, the expression for a_n becomes an instantaneous expression, valid only at the instant when the velocity is v. If the path is curvilinear but not circular, the value of r in the expression is the distance from the point at which velocity is v, to the center of curvature at the instant, or the instantaneous radius of curvature.

An equivalent expression for normal acceleration may be derived in terms of the angular velocity of the moving radius.

Since
$$a_n = \frac{v^2}{r}$$

and
$$v = \omega r$$

Then
$$a_n = \frac{\omega^2 r^2}{r} = \omega^2 r.$$

The value of ω and of r are instantaneous values if the angular velocity or the radius of curvature (or both) are changing quantities.

At any instant, then, there is a normal acceleration which is dependent only upon the instantaneous value of the velocity and of the radius of curvature. This acceleration is present at such an instant whether the speed later changes or remains constant.

If, however, the velocity does change in magnitude, such a change must be in the direction of the velocity, which is tangential to the path.

We shall call this type of acceleration the *tangential component* of acceleration, a_t. Its magnitude may be obtained from the basic definition of acceleration,

$$a_t = \frac{dv}{dt}.$$

To evaluate this term, it is convenient to turn to the relationship between linear and angular accelerations which will be discussed in the next section.

103. Linear and Angular Acceleration. *Angular acceleration, α, is the rate of change of angular velocity of a line with respect to time,* or the time rate of change of angular velocity.

Then
$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

The tangential component of linear acceleration

$$a_t = \frac{dv}{dt}$$

may be most readily evaluated by noting that since

$$v = \omega r$$

$$a_t = \frac{d\,(r\omega)}{dt} = r\,\frac{d\omega}{dt} = r\alpha.$$

This component of acceleration depends then upon the angular acceleration of the moving line, as well as on the instantaneous or permanent radius of curvature.

104. Resultant Acceleration. A complete investigation of the linear acceleration of a point moving with curvilinear motion, and with variable speed, necessitates establishing two vector quantities, normal and tangential accelerations.

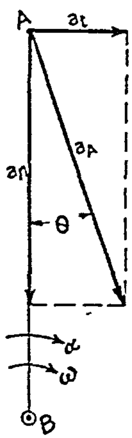


FIG. 299

The resultant acceleration is the vector sum of these two, as in Fig. 299.

A particle at A on radius AB is constrained to rotate about center B. Line AB is 2 inches long, and has angular velocity ω of 3 revs. per sec., clockwise, and angular acceleration α of 20 revs. per sec. per sec., clockwise. The normal component

$$a_n = {}^*\omega^2 r = (3 \times 2\pi)^2\, 2 = 711 \text{ inches per sec. per sec.}$$

The tangential component,

$$a_t = \alpha r = (20 \times 2\pi)\, 2 = 251 \text{ inches per sec. per sec.}$$

Then the resultant acceleration of point A is $a_A = a_t \nrightarrow a_n$

or
$$a_A = \sqrt{\overline{a_t}^2 + \overline{a_n}^2} = 754 \text{ inches per sec. per sec.}$$

* Note that ω must be expressed in radians per unit of time.

The resultant is inclined so that it makes an angle

$$\theta = \tan^{-1} \frac{251}{711} = 19.4°$$

with the radius AB. The sense of the resultant is determined from the senses of the two components.

PROBLEMS

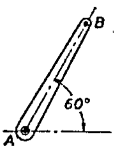
392. The angular velocity of crank AB is 2 radians per min. clockwise, and its angular acceleration is 4 radians per min. per min., counter-clockwise in the position shown. Determine the resultant acceleration of point B. $AB = 4$ in.

Ans. 22.6 in. per min. per min.

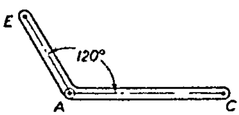
393. Determine the resultant acceleration of points C and E if the rocker arm CAE rotates about fixed axis A, and has an angular velocity of 6 radians per min., clockwise, and an angular acceleration of 30 radians per min. per min., clockwise, in the position shown. $AC = 7.5$ in.; $AE = 4.5$ in.

394. An eccentric wheel rotates about axis O and has an angular velocity of 4 radians per sec. counter-clockwise, and an angular acceleration of 12 radians per sec. per sec. clockwise in the position shown.

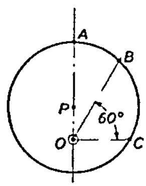
Determine the resultant accelerations of points A, B, and C. Diameter of wheel = 8 in. with center at P. $OP = 2.3$ in.



PROB. 392



PROB. 393



PROB. 394

105. Absolute and Relative Accelerations. The basic statement for all concepts of absolute and relative motion, as outlined in Theorem III, applies to accelerations.

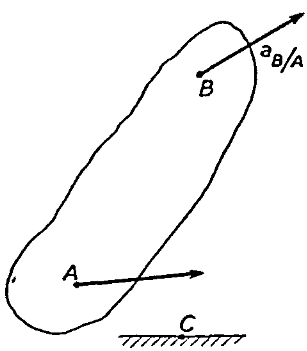


FIG. 300

The *absolute acceleration* of a particle then is the sum of the acceleration of that particle relative to any other particle and the absolute acceleration of the other particle.

If, as in Fig. 300, the acceleration of point B relative to point A is known as $a_{B/A}$, and the absolute acceleration of point A (that is, its acceleration relative to a point C on the earth's surface) is known as a_A, the absolute acceleration of B is a_B, the vector sum of a_A and $a_{B/A}$, or $a_B = a_A \nrightarrow a_{B/A}$ (Fig. 301).

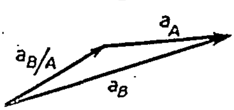


FIG. 301

This method of finding B's absolute acceleration may seem, at first, to be indirect; in reality, it forms the most effective means of exploring for acceleration relationships.

The principle is equally valid when applied to components of acceleration.

If we obtain the orthogonal component of A's absolute acceleration in the direction of AB, we have, in either Fig. 302 or 303, Aa_3. If we obtain the orthogonal component in this same direction of B's acceleration relative to A, we have Bb_3.

It will be seen in Fig. 303 that Bb_4, which is the orthogonal component of B's absolute acceleration in direction AB, is the vector sum of these orthogonal components Aa_3 and Bb_3.

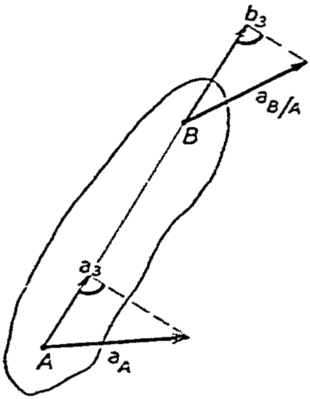


Fig. 302

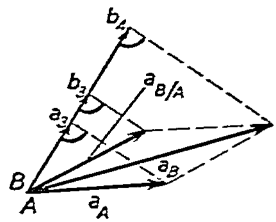


Fig. 303

This conclusion may be stated as follows:

The orthogonal component of the absolute acceleration of a particle in a given direction is equal to the sum of the orthogonal component in that direction of the acceleration of that particle relative to another particle, plus the orthogonal component in the same direction of the absolute acceleration of the second particle.

106. Acceleration in Pin-Connected Bodies. A typical analysis of acceleration relationships is involved in the case of a four-bar linkage shown in Figs. 304 and 306.

It will be assumed that the dimensions of all members are known, and that the kinematic properties—velocity and acceleration—of AB are also known.

An orderly and effective solution of the problem will require a complete knowledge of the angular and linear velocities of the several bodies and their particles.

Figure 304 is the velocity analysis. This repeats the application of principles with which we are now familiar from the earlier work in velocity. Bb' is obtained as the resultant (and absolute) velocity of point B, and Cc' is the resultant absolute velocity of point C.

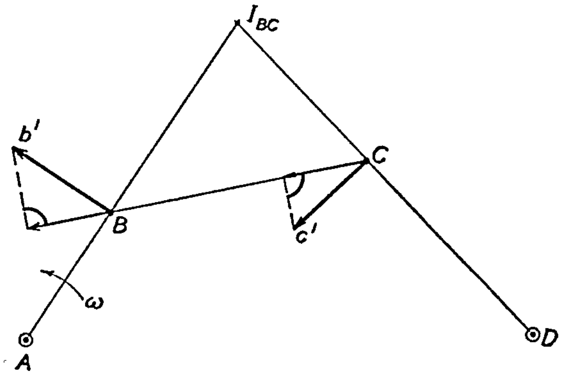


FIG. 304

The angular velocity of body AB, ω_{AB}, is given in the original data, and $Bb' = \omega_{AB} \cdot AB$.

The angular velocity of body BC may be obtained by dividing linear velocity Bb' by radius $I_{BC}B$, the distance to the instantaneous axis of velocities, or Cc' may be divided by $I_{BC}C$ to give ω_{BC}.

The linear velocity of C, Cc', may be divided by radius CD, to obtain the angular velocity of CD, ω_{CD}.

The velocity of C relative to B, $v_{C/B}$, is obtained as a vector difference in Fig. 305, and must be perpendicular to $BC \cdot v_{C/B} = v_C \rightarrow v_B$. These values are all *instantaneous* values. They will change continuously as the crank AB is rotated to new positions, and in each position of the driving crank, a solution of velocities, valid for that instant only, may be obtained by the methods outlined.

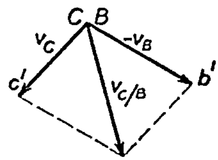


FIG. 305

A complete description of linear and angular velocity relationships in the bodies comprising the mechanism is now available to initiate the attack upon accelerations.

The linear acceleration of point B is a_B (Fig. 306).

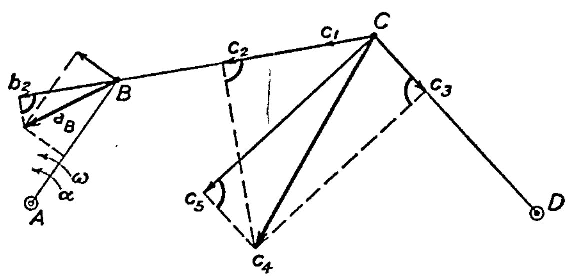


FIG. 306

The components of this acceleration are

$$a_t = \alpha_{AB} \cdot AB$$

$$a_n = (\omega_{AB})^2 \cdot AB$$

and

$$a_B = \sqrt{\overline{a_t}^2 + \overline{a_n}^2}.$$

The acceleration of point C may be now found by using the principle of absolute and relative accelerations.

The component acceleration of point C relative to point B in the direction BC is a normal component of acceleration. This may be appreciated fully by noting that the only motion which C may have, relative to B, is a motion of pure rotation about B. Two points on the same rigid body *may have no relative velocity in the direction connecting them.* Therefore any velocity which C may have, relative to B, must be in a direction normal to BC. Then the relative velocity relationship between C and B is the same in nature as would be obtained if B were considered, for the instant only, an axis of rotation, and C were moving in a circular path about B.

$$v_{C/B} = v_C \rightarrow v_B.$$

This appraisal of the nature of C's motion is one which has been made before (see Art. 49). When a rigid body, like BC, has plane motion, the absolute motion of any point, like C, may be broken up into two contributing elements—the motion of C relative to any other point of the body, like B, and the absolute motion of the second point.

The normal component of C's acceleration relative to B is then $a_n = \dfrac{v^2}{r}$ where the velocity, v, is the relative velocity of C with respect to B, which was obtained in the analysis of velocities as $v_{C/B}$; and r is the distance BC.

C must also have a tangential component of relative acceleration with respect to B. This component is not needed in the present analysis, but attention is called to it so that the normal component of C's acceleration relative to B will not be interpreted as a *resultant* acceleration.

The normal component of acceleration of C relative to B just obtained is called Cc_1 in Fig. 306.

$c_1c_2 = Bb_2$, which is the orthogonal component of B's absolute acceleration in direction BC, is now added to Cc_1.

The vector sum of Cc_1 and $c_1c_2 = Cc_2$ is the orthogonal component of C's absolute acceleration in direction BC.

The principles of Theorems I and II of orthogonal components demand that in finding a resultant vector we must have either (1) an orthogonal component and a direction of resultant, or (2) two orthogonal components.

One orthogonal component (Cc_2) of C's absolute acceleration is now

determined, but no basis of prediction of its inclination is readily available and another orthogonal component must therefore be found.

The absolute velocity of C is already known as Cc' (of Fig. 304) from the previous velocity analysis. Then C has a normal component of acceleration relative to D, $a_n = \frac{v^2}{r}$ where v is Cc' and r is CD. This is also an orthogonal component of C's absolute acceleration in direction CD, for D, which is *fixed*, has zero orthogonal component of absolute acceleration in this or any other direction.

This orthogonal component of acceleration is recorded as Cc_3 in Fig. 306, and is combined with orthogonal component Cc_2 to find the resultant Cc_4, which is the absolute linear acceleration of point C.

The angular acceleration of crank CD may now be found. Since Cc_4 is an absolute acceleration, its orthogonal component (Fig. 306) perpendicular to CD, Cc_5, must be the tangential component of C's acceleration with respect to fixed point D.

Then dividing Cc_5 by CD the angular acceleration of crank CD is obtained.

$$\alpha_{CD} = \frac{a_t}{r} = \frac{Cc_5}{CD}.$$

The connecting rod BC has angular acceleration α_{BC} which may be found by considering again the fact that, *relative to B*, radius BC has a motion of pure rotation at the instant with B serving as a center of rotation.

Then points along BC have tangential accelerations relative to B which are perpendicular to BC and equal in magnitude to the angular acceleration of BC, α_{BC}, times the radius from B to the point.

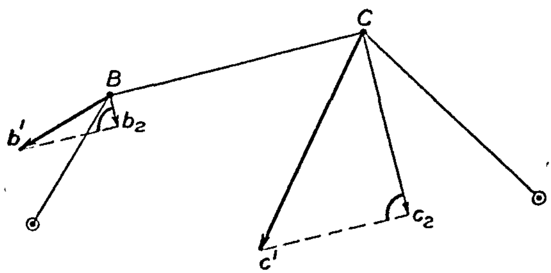


Fig. 307

This relative motion may be utilized in finding α_{BC}. In Fig. 307 point B has absolute acceleration Bb', and an absolute orthogonal component of acceleration Bb_2, perpendicular to BC.

Point C whose absolute acceleration is Cc' has an absolute orthogonal component Cc_2 perpendicular to BC.

The vector difference between the orthogonal components Cc_2 and $Bb_2 = Cb_2$ of Fig. 308 is the *relative acceleration of C with respect to B* in a direction perpendicular to BC, or the tangential acceleration of C relative to B. If this tangential acceleration is divided by BC, the quotient will be the magnitude of the angular acceleration of connecting rod BC.

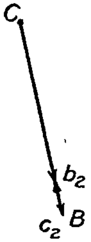


FIG. 308

This analysis of acceleration in the four-bar linkage has rested upon bases which may be stoutly defended. We have used the definitions of acceleration and of component accelerations, the theorems of absolute and relative motions, and the theorems of orthogonal components.

The analysis may be conveniently summarized in equation form.

$$a_C = a_{C/B} \mathrel{+\!\!\!\!\rightarrow} a_B$$

or

$$(a_C)_t \mathrel{+\!\!\!\!\rightarrow} (a_C)_n = (a_{C/B})_t \mathrel{+\!\!\!\!\rightarrow} (a_{C/B})_n \mathrel{+\!\!\!\!\rightarrow} (a_B)_t \mathrel{+\!\!\!\!\rightarrow} (a_B)_n.$$

There are available in treatises on kinematics, specialized "constructions" which will also solve the problem of obtaining accelerations. "Construction" methods are objectionable in that they may be used, like other formulas, without basic understanding; and they may suffer, as do all formulas, from abuse. The one advantage of construction methods rests in the reduction of the number of lines which must be drawn to accomplish a solution. Such geometrical devices, in short-cutting the stages of solution, defeat the purpose of the student of acceleration who can only feel secure in an analytical exploration towards an objective when he has truly mastered each stage.

The mastery of the acceleration principles involved in the analysis of point C of the connecting rod enables one to attack confidently the problem of finding the acceleration of any point on that rigid body. For example, the acceleration of any point on rigid body BC may now be determined. It will be observed, as the analysis is developed, that no new weapons of attack must be supplied—this basic method is thoroughly penetrating and will be adequate for a complete solution. In other examples which will follow, an equipment of definitions and basic theorems will again be found to be powerful enough to accomplish the solution; acceleration problems, when faced with such equipment, yield readily and satisfactorily, and mastery of fundamental principle will avoid the pitfalls of formula substitution.

It is assumed in the problem shown in Fig. 309 that the investigation has been carried through the stage shown in Fig. 307 and that the kinematic properties, α_{BC} and ω_{BC}, of body BC are known. The absolute acceleration of point S, any point on the line BC, is to be determined.

Every point on line BC has a tangential component which is perpendicular to BC, and of magnitude equal to αr, where α is α_{BC} and r is the radius from a center, in this case point R, of the tangential components. Then a line joining b_2 and c_2 establishes s_2, and the tangential component Ss_2 of the absolute acceleration of S has been found.

Ss_1, the acceleration of S relative to B in the direction BS, is next determined as $\omega^2 r$, where ω is ω_{BC} and r is the length of BS. If $s_1s_3 = Bb_3$, the orthogonal component of B's absolute acceleration is added to Ss_1, the

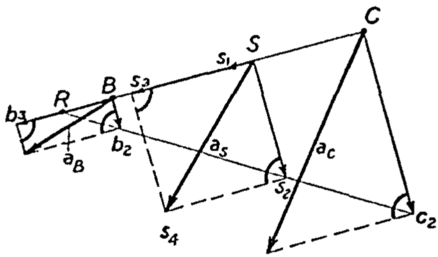


FIG. 309

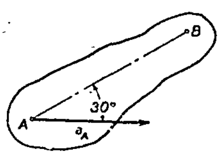
sum Ss_3 is the orthogonal component of the absolute acceleration of S in the direction BS.

Now two orthogonal components of S's absolute acceleration are known, and their resultant is Ss_4, the absolute acceleration of point S.

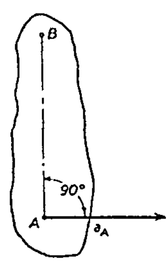
$$a_s = Ss_4 = Ss_2 \nrightarrow Ss_3.$$

PROBLEMS

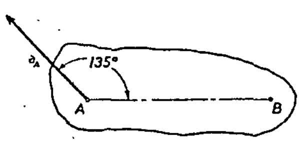
Problems 395–398. The body shown has constant absolute angular velocity, $\omega = 1$ radian per sec. The absolute acceleration of point A is a_A, which is shown in direction and has magnitude = 4 in. per sec. per sec. Determine the resultant acceleration of B relative to A, and the absolute acceleration of B. $AB = 6$ in.



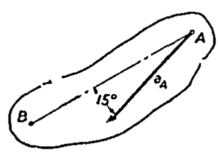
PROB. 395



PROB. 396

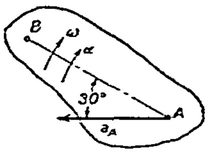


PROB. 397

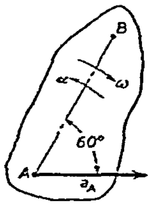


PROB. 398

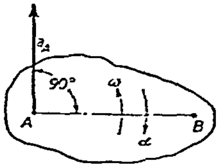
Problems 399–402. The body shown has angular velocity $\omega = 2$ radians per sec. and angular acceleration, $\alpha = 5$ radians per sec. per sec. (senses shown). The absolute acceleration of A is $a_A = 10$ in. per sec. per sec. $AB = 5$ in. Determine the resultant acceleration of B relative to A, and the absolute acceleration of B.



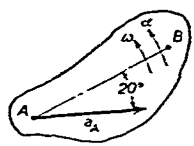
Prob. 399



Prob. 400

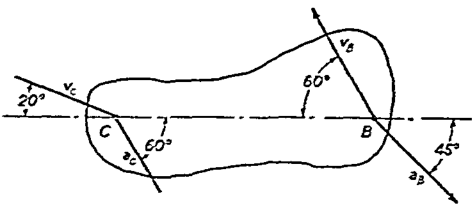


Prob. 401

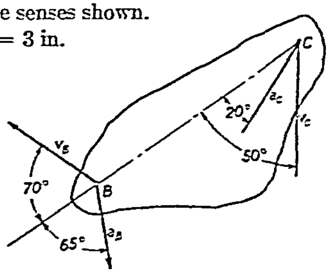


Prob. 402

Problems 403–404. The inclinations of the absolute velocities and accelerations of two points, B and C, are shown. The absolute speed of B is 2 in. per sec., and the absolute acceleration of B is 2 in. per sec. per sec., in the senses shown. Determine the absolute acceleration of point C. $BC = 3$ in.



Prob. 403

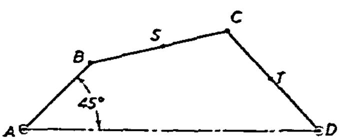


Prob. 404

405. Determine the angular velocity and the angular acceleration of the body described in Problem 403.

406. Determine the angular velocity and the angular acceleration of the body described in Problem 404.

407. The driving crank has a constant angular velocity of 1 radian per sec. clockwise.



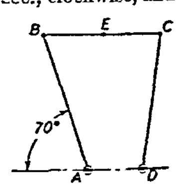
Prob. 407

Determine the absolute acceleration of point C, for the position shown. $AB = 4.5$ in.; $BC = 6.5$ in.; $CD = 6.3$ in.; $AD = 13.8$ in.

Ans. AC (magnitude) = 10 in. per sec. per sec.

408. Using the data of Problem 407, determine the absolute accelerations of points S and T. $CS = 3.0$ in.; $CT = 3.0$ in.

409. Solve Problem 407 if AB has angular velocity of 2 radians per sec., clockwise, and angular acceleration of 3 radians per sec. per sec. clockwise in the position shown.



Prob. 411

410. Determine the angular acceleration of bodies BC and CD of Problem 409.

411. Crank AB has angular velocity of 4 radians per sec., clockwise, and angular acceleration of 10 radians per sec. per sec. counterclockwise. Determine the absolute accelerations of points C and E, for the position shown. $AB = 6.9$ in.; $BC = 6.0$ in.; $CD = 6.5$ in.; $AD = 2.9$ in.; $CE = 3.0$ in.

412. Determine the angular acceleration of crank CD of Problem 411.

413. Using the mechanism of Problem 150, determine the absolute acceleration of point E in the position shown, if the crank containing point A has a constant absolute angular velocity of 100 r.p.m. clockwise.

414. Using the mechanism of Problem 151, determine the absolute acceleration of point E if crank BC has an angular velocity of 1 radian per sec. clockwise, and an angular acceleration of 1 radian per sec. per sec. counter-clockwise, for the position shown.

415. Using the Peaucellier's linkage given in Problem 135, determine the absolute acceleration of point C in the position shown if the crank BF has a constant angular velocity of 1 radian per min.

107. Acceleration of the Instantaneous Axis of Velocities. The instantaneous axis of velocities is a point of zero velocity. We have already noted in Art. 47 that the particle of a rigid body which serves at a given instant as its instantaneous axis will at any later instant lie in such a position on the body that it will now have velocity, and a different particle of the rigid body will have become the point of zero velocity. Then there has been change in the velocity of the particle which originally served as instantaneous axis. A change in velocity involves acceleration. A particle serving as an instantaneous axis of velocities has, therefore, an acceleration.

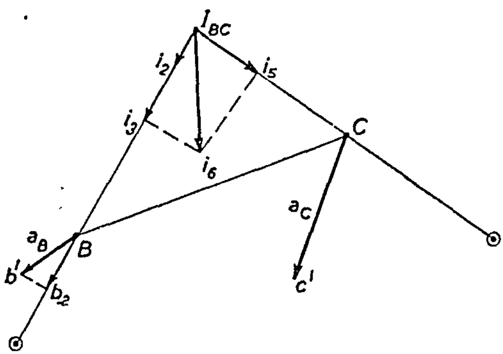


FIG. 310

The absolute accelerations of B and C (Fig. 310) are known, and are Bb' and Cc' respectively.

As before, ground may be broken by a velocity study in which we should determine the velocity of I_{BC} relative to B and its velocity relative to C.

The normal component of acceleration of I_{BC} relative to B will again be $a_n = \frac{v^2}{r}$ where v is the velocity of I_{BC} relative to B, and r is the distance $I_{BC}B$. This acceleration component is $I_{BC}i_2$ in the figure. To it must be added $i_2i_3 = Bb_2$, the orthogonal component in direction $I_{BC}B$ of B's absolute acceleration.

Then $I_{BC}i_3$, the vector sum of these components, is one orthogonal component of I_{BC}'s absolute acceleration, in the direction $I_{BC}B$.

$I_{BC}i_5$ is obtained similarly by relating I_{BC} to point C, and two orthogonal components are now available, which may be combined by application of Theorem II to yield a resultant $I_{BC}i_6$, which is the absolute acceleration of the instantaneous axis of velocities, I_{BC}.

108. The Instantaneous Center of Accelerations is defined as the point on an accelerating body which has, at a given instant, *zero absolute acceleration.*

If a body, like that of Fig. 311, has at a given instant angular velocity, ω, and angular acceleration, α, which are both known, a basis of method is available which may be used in a search for the instantaneous center of accelerations.

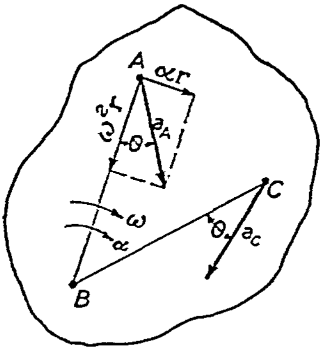


Fig. 311

A and B are two particles, and the distance between them is r.

Then the acceleration of A relative to B has a tangential component

$$a_t = \alpha r$$

and a normal component

$$a_n = \omega^2 r.$$

The angle θ is $\tan^{-1}\left[\frac{\alpha r}{\omega^2 r} = \frac{\alpha}{\omega^2}\right]$ and forms the angle between radius AB and the resultant acceleration of A relative to B.

If B were a point of zero absolute acceleration, then θ would be the angle between radius AB and the *resultant absolute* acceleration of point A.

The same angle θ will then be formed by the radius BC and the *resultant* absolute acceleration of C, where C is any point on the rigid body.

Then if we know the absolute acceleration of two points on a rigid body, we may determine the instantaneous center of accelerations.

A specific application will serve to clarify the procedure. On the body of Fig. 312 particle A has, at a given instant, absolute acceleration Aa'. At the same instant the absolute acceleration of particle C is Cc'.

Then the acceleration of A relative to C is Aa_2, obtained as the vector difference between Aa' and Cc' $(a_{A/C} = a_A \rightarrow a_C)$. The angle between Aa_2 and line AC will be θ.

The component of Aa_2 in the direction of AC is Aa_3, which is a normal component of the acceleration of A relative to C.

Then $Aa_3 = (\omega_{AC})^2 \times AC$.

Aa_4 is the component of Aa_2 perpendicular to AC, and is therefore the tangential component of the acceleration of A relative to C.

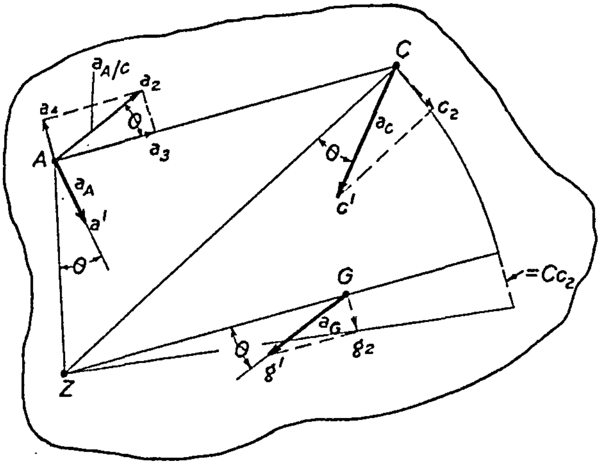


Fig. 312

Then $$Aa_4 = \alpha_{AC} \times AC.$$

The angle $\theta = \tan^{-1}\left[\dfrac{\alpha_{AC} \times AC}{\omega_{AC}^2 \times AC} = \dfrac{\alpha_{AC}}{\omega_{AC}^2}\right].$

Since α and ω are, at any instant, the same for all lines of the body, this angle θ will be the angle between the absolute acceleration of any point and the line joining that point and the instantaneous center of accelerations of the body.

If, now, as in Fig. 312, we draw a line AZ making an angle θ with Aa', the absolute acceleration of A, and another line CZ, making an angle θ with Cc', the absolute acceleration of C, AZ and CZ will intersect at point Z, the *instantaneous center of accelerations* for the body containing A and C. There are two lines which, like AZ, will make an angle θ with absolute acceleration Aa'; and there are two which, like CZ, will make an angle Z with Cc'. The correct pair may be isolated by trial of the four possibilities noting that Z must be so located that the senses of the normal and tangential components of the absolute accelerations of A and C are consistent.

The acceleration of point G, another point on the body, may now be found by noting that it has an inclination of θ with line ZG. To establish its magnitude one orthogonal component is necessary; for example,

$$Gg_2 = a_t = \alpha_{AC} \times ZG.$$

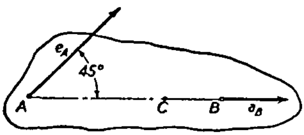
The value of α_{AC} became available when Aa_2, the acceleration of A relative to C, was established, for the orthogonal component of Aa_2 at right angles to AC (Aa_4) is the tangential component of A's acceleration relative to C, and is therefore

$$Aa_4 = a_t = \alpha_{AC} \cdot AC,$$

then
$$\frac{Aa_4}{AC} = \alpha_{AC}.$$

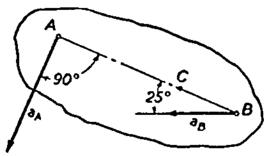
Gg_2 may also be established, graphically, by proportion from Cc_2, as in the figure.

PROBLEMS

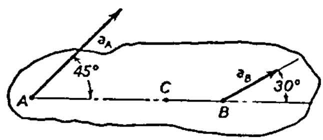


Prob. 416

Problems 416–418. The magnitude of the absolute acceleration of point A is 4 in. per sec. per sec., and that of B is 2 in. per sec. per sec. The inclinations and sense are shown. Locate the instantaneous axis of accelerations of the body, and determine the absolute acceleration of point C. $AB = 6$ in.; $AC = 4.2$ in.



Prob. 417



Prob. 418

419. Locate the instantaneous axis of accelerations of body BC of Problem 411, and determine the absolute acceleration of point E.

109. Acceleration in Rolling Contact. The cylinder shown in Fig. 313 is in pure rolling contact with a fixed track, and has angular velocity, ω, which is constant. The acceleration of any point on the cylinder may be found by applying, as in previous cases, the basic theorems.

For example, I, the point of contact with the track, has zero velocity, and is the instantaneous axis of velocities of the cylinder.

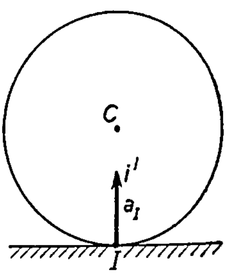


Fig. 313

The velocity of I relative to C, the center of the cylinder, is $v_1 = \omega \times IC$, and the acceleration of I relative to C in the direction connecting I and C is a normal component of relative acceleration. This is $Ii' = \dfrac{v_1^2}{IC}$ or $Ii' = \omega^2 \cdot IC$.

This normal component is the *resultant* acceleration of I relative to C, for with ω announced as constant, $\alpha = 0$.

To find the absolute acceleration of I we must add, to its acceleration relative to C, the absolute acceleration of C, which in this case is zero, for point C has rectilinear motion with constant speed. Ii' is the absolute acceleration of I.

If the cylinder has an angular acceleration, α (Fig. 314), the acceleration of I relative to C will have a tangential component $Ii_3 = a_t = \alpha \cdot IC$; and a normal component $Ii_2 = a_n = \omega^2 \cdot IC$. C will also have an absolute acceleration Cc' parallel to the track and equal to $\alpha \cdot IC$. $Cc' = Ii_3$. Then the absolute acceleration of I will be Ii_2, the sum of Ii' and Cc', in the direction of radius IC, as it was when the rolling body had constant angular velocity.

$$a_I = a_{I/C} \mathrel{+\!\!\!\!\rightarrow} a_C.$$

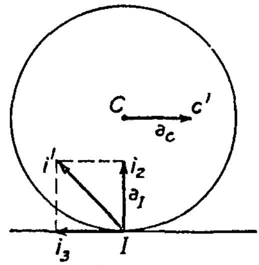


Fig. 314

This conclusion may be verified if we investigate the motion from another approach.

The path of the particle of the cylinder which is, at the instant shown, at I, is cycloidal. This path is illustrated in Fig. 315. The cycloidal path is normal to the track at the instant of contact, and I must therefore have a resultant absolute acceleration normal to the track.

Another example of an acceleration study in pure rolling contact is illustrated in Fig. 316. In this epicyclic wheel train the arm, A, has absolute angular velocity ω_A and absolute angular acceleration α_A. The driving wheel has constant angular velocity ω_D. The resultant absolute acceleration of point f on the follower is ff'. This is determined from the given data concerning the arm, for point f is a point on the arm as well as on the follower.

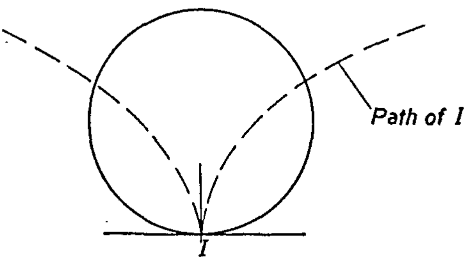


Fig. 315

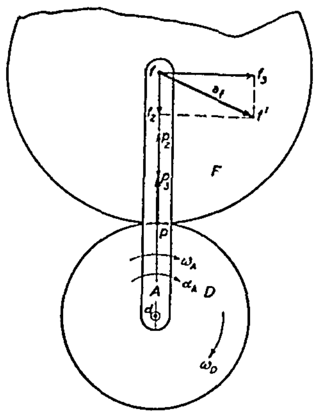


Fig. 316

$$ff_2 = \omega_A{}^2 \times df$$

$$ff_3 = \alpha_A \times df$$

$$ff' = \sqrt{\overline{ff_2}^2 + \overline{ff_3}^2}$$

The point of contact, p (on the follower), has a normal component of acceleration pp_2 *relative to* f which may be determined as follows: The absolute angular velocity of the follower ω_F is obtained as in any epicyclic train (see Art. 87). Then $pp_2 = (\omega_F)^2 \cdot fp$. To pp_2 we next add p_2p_3 equal to the component of f's absolute acceleration in direction fp. The sum of pp_2 and p_2p_3 is pp_3 which is the orthogonal component in direction fp of p's absolute acceleration. pp_3 is also the resultant absolute acceleration of point p, for point p has no horizontal component of acceleration. This statement may be confirmed by noting that point p on the driver has no horizontal component of acceleration (ω_D constant) and point p on the follower is in pure rolling contact with it; and that the path of point p is epicycloidal, with a cusp at the point of contact.

The acceleration of any other point on the follower may be obtained now that the absolute acceleration of two points on that body have been determined.

PROBLEMS

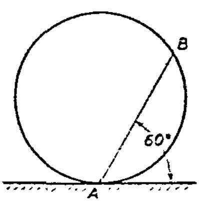
420. The 6 in. diameter cylinder is in pure rolling contact with the fixed track at A, and has a constant angular velocity of 1 radian per sec. counter-clockwise. Determine

(a) The acceleration of point B, relative to point A.

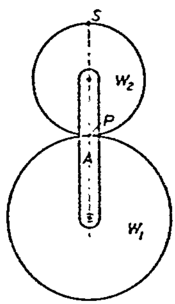
(b) The absolute acceleration of point A.

(c) The location of the instantaneous axis of accelerations.

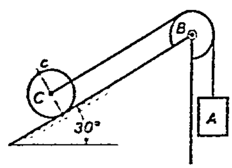
Ans. (a) 5.20 in. per sec. per sec. (b) 3.0 in. per sec. per sec.



Prob. 420



Prob. 422



Prob. 424

421. Solve Problem 420 if the cylinder has an absolute angular velocity of 1 radian per sec. clockwise and an absolute angular acceleration of 1 radian per sec. per sec. clockwise.

422. In the epicyclic wheel train, W_1 has a 7.5 in. dia., and W_2 has a 5.2 in. dia. The wheels are in pure rolling contact. W_1 has constant angular velocity of 1 radian per sec. clockwise. The arm, A, has angular velocity of 2 radians per sec. clockwise, and angular acceleration of 3 radians per sec. per sec. clockwise for the position shown. Determine the acceleration of point P, the point of contact on W_2.

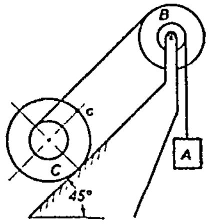
423. Locate the instantaneous axis of accelerations of wheel W_2 in Problem 422 and determine the acceleration of point S.

424. A has an absolute velocity of 5 ft. per sec. and an absolute acceleration of 10 ft. per sec. per sec., both downward, in the position shown. The cord joining A with the

center of C passes over pulley B, which is mounted on a fixed axis. C is in pure rolling contact with the inclined plane. The position of cord from B to C is parallel to the plane. Determine the absolute acceleration of point c on the rolling cylinder. Dia. of $C = 4$ ft. *Ans.* 23.6 ft. per sec. per sec.

425. A has an absolute velocity of 6 ft. per sec. and an absolute acceleration of 14 ft. per sec. per sec., both downward. A two-step pulley B is mounted on a fixed axis. Inner dia. = 1.5 ft. Outer dia. = 3 ft.

The cylinder C is in pure rolling contact with the inclined plane. C has inner dia. = 1.8 ft., and outer dia. = 4 ft. The cord from B to C is parallel to the plane. Determine the absolute acceleration of point c on the rolling cylinder.



PROB. 425

110. Acceleration in Sliding Contact.

Fixed Guides. In Fig. 317 AB is a driving crank, and BC a connecting rod which is pinned to AB at B, and to a block at C which is constrained to

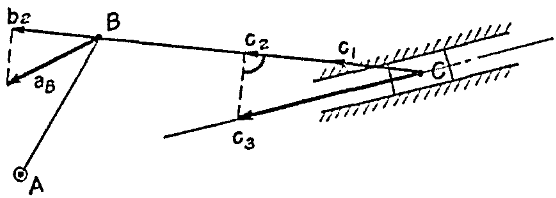


FIG. 317

move in fixed guides. The dimensions of the links and the kinematic properties of the driving crank are known.

The orthogonal component of C's absolute acceleration in direction BC is Cc_2, which is the vector sum of the normal component of C's acceleration relative to B, in direction BC, Cc_1, and the orthogonal component of B's absolute acceleration in the same direction, $c_1c_2 = Bb_2$.

The inclination of C's absolute acceleration is known since it must be parallel to the sliding surfaces.

One orthogonal component and inclination of the resultant vector are then known and the application of Theorem I yields the resultant Cc_3, which is the absolute acceleration of point C, and, since the block is moving with pure translation, is the absolute acceleration of any point on the block.

In Fig. 318 a cam is placed in contact with a pointed follower. The identification of the equivalent linkage is shown. Point B is the center of curvature of the cam surface at the point of contact. The equivalent linkage has the same relative size of members only when the curvature has radius BC, and the center of curvature is at B. As the cam surface changes, the identification of the equivalent linkage in each new position must be prepared by locating the center of curvature for the cam surface in that position.

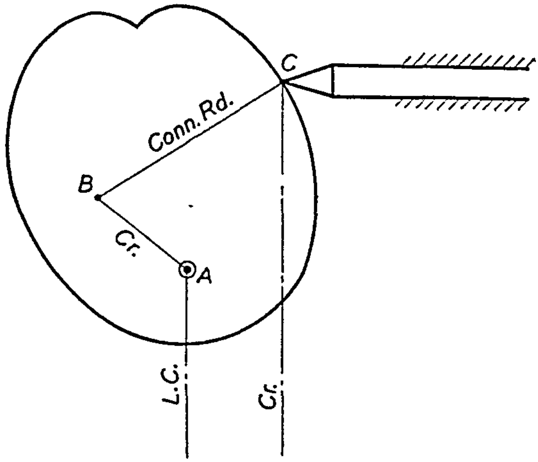


Fig. 318

The identification reveals that the equivalent linkage is of the crank and connecting-rod form, and the acceleration analysis is identical with the procedure employed in that case.

Moving Guides. When the sliding contact takes place in guides carried by a moving body, the acceleration analysis becomes somewhat more complex. When, as in the velocity analyses, the equivalent four-bar linkage background may be readily identified, the acceleration analysis which has been developed for the four-bar linkage may be effectively applied.

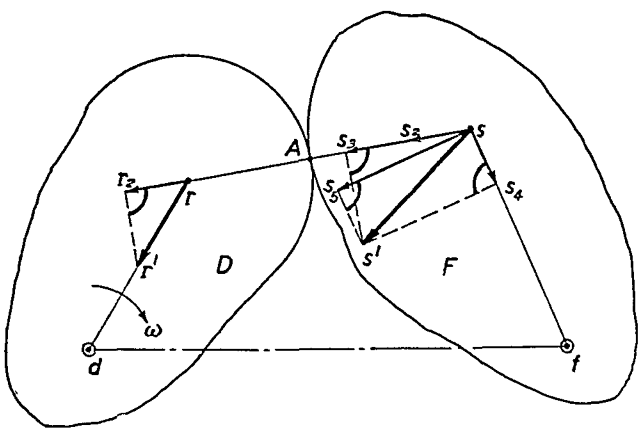


Fig. 319

The driver and follower of Fig. 319 are in sliding contact at point A. At this instant, both curved surfaces must have a common normal.

If the center of curvature of the contacting surface on D is, for the in-

stant shown, at point r, and that of F is at point s, the line rs is the common normal.

Point r is rotating about fixed axis d, and point s about fixed axis f.

Then the motion is, for the instant, directly equivalent to that of a four-bar linkage, with

$$\begin{aligned} \text{Driving crank} &= dr \\ \text{Connecting rod} &= rs \\ \text{Follower crank} &= sf \\ \text{Line of centers} &= df. \end{aligned}$$

Now that the equivalent linkage has been identified, the acceleration investigation will parallel that of any pin-connected four-bar linkage (see Art. 106). For example, let it be required that the angular acceleration of F be determined for a constant angular velocity, ω_D, of the driving crank.

The absolute acceleration of point r on D is

$$rr' = (\omega_D)^2 \times dr.$$

The orthogonal component of rr' along rs is rr_2.

The acceleration of s relative to r has a normal component in direction $sr = ss_2$. This is determined as in other cases by a preparatory velocity study. To ss_2 must be added $s_2s_3 = rr_2$. The sum, ss_3, is the orthogonal component in the direction sr of the absolute acceleration of point s.

Another orthogonal component of the absolute acceleration of s is available from the velocity study—this is ss_4, the component of s's acceleration relative to f.

ss_3 and ss_4 satisfy the demands of Theorem II and yield ss', which is the absolute acceleration of point s. From this ss_5, the orthogonal component of ss' perpendicular to sf, may be found. Since ss_5 is perpendicular to sf, it is the tangential component of the acceleration of point s relative to f, and

$$\alpha_F = \frac{ss_5}{sf}.$$

This method of attack has depended upon identification of the equivalent four-bar linkage, which in turn became readily available when the centers of curvature of the contacting arcs were available.

Another instance in which identification of equivalent linkage background leads to a direct solution of an acceleration problem is illustrated in Fig. 320.

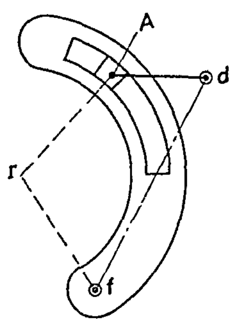


Fig. 320

In this swinging-block mechanism, the sliding block is placed in a circular slot whose center of curvature is at point r.

Then the equivalent four-bar linkage is again identified by searching for the four links which correspond to basic definition as members of a four-bar linkage.

The driving crank = dA
Connecting rod = Ar
Follower crank = rf
Line of centers = df.

The acceleration analysis follows the usual procedure for the four-bar linkage.

The foregoing examples illustrate the very penetrating value of the equivalent linkage method of attack.

The method, while theoretically available, is, however, not always as readily applied as in those cases where a finite radius of curvature of the sliding surface leads to direct solution of the acceleration problem.

For example, in cases where the equivalent linkage identification reveals that some members of the linkage are infinitely long, such an approach is, in general, inadequate. One exception to this statement has been discussed under "Fixed Guides" on page 293.

Another case where the equivalent linkage method is at a disadvantage when infinite equivalent links are encountered has been noted in the example of the Scotch Yoke as an application of simple harmonic motion.

Figure 321 illustrates a case in which identification of the equivalent

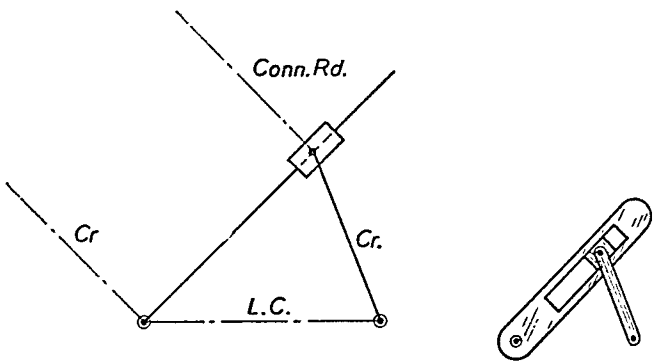


FIG. 321

linkage fails to lead to a direct solution of the acceleration problem, and we must develop other methods of attack.

The problem may be viewed in its simplest form by considering a particle, A, of Fig. 322. The motion of the particle is described as follows:

Particle A is moving outward (away from O) on radius R at the same time that radius R turns about the origin O with angular velocity, ω, and angular acceleration, α.

The problem is to find the absolute acceleration of A.

In previous studies of acceleration, radii, like R, were of fixed length.

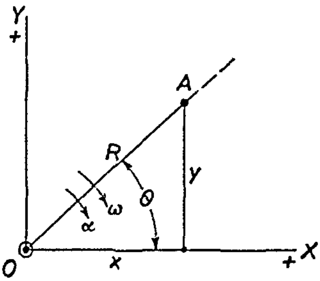


FIG. 322

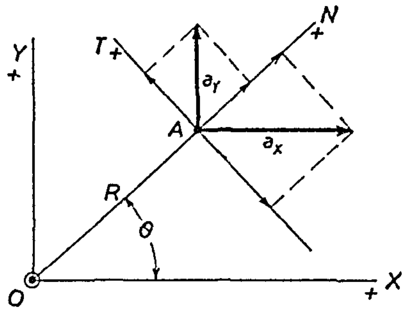


FIG. 323

We now are faced with the problem of accelerations when the radial distance is *varying*.

Expressions for the X and Y components of the absolute acceleration of A may be obtained by differentiation.

$$x = R\cos\theta \qquad \text{and} \qquad y = R\sin\theta$$

$$v_x = \frac{dx}{dt} = \frac{dR}{dt}\cos\theta - R\sin\theta\frac{d\theta}{dt}$$

$$a_x = \frac{d^2x}{dt^2} = \frac{d^2R}{dt^2}\cos\theta - \frac{dR}{dt}\sin\theta\frac{d\theta}{dt} - \frac{dR}{dt}\sin\theta\frac{d\theta}{dt}$$

$$- R\cos\theta\left(\frac{d\theta}{dt}\right)^2 - R\sin\theta\frac{d^2\theta}{dt^2}$$

$$= \cos\theta\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right] - 2\frac{dR}{dt}\sin\theta\frac{d\theta}{dt} - R\sin\theta\frac{d^2\theta}{dt^2}$$

$$v_y = \frac{dy}{dt} = \frac{dR}{dt}\sin\theta + R\cos\theta\frac{d\theta}{dt}$$

$$a_y = \frac{d^2y}{dt^2} = \frac{dR}{dt}\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{d^2R}{dt^2} + \frac{dR}{dt}\cos\theta\frac{d\theta}{dt}$$

$$- R\sin\theta\left(\frac{d\theta}{dt}\right)^2 + R\cos\theta\frac{d^2\theta}{dt^2}$$

$$= \sin\theta\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right] + 2\frac{dR}{dt}\cos\theta\frac{d\theta}{dt} + R\cos\theta\frac{d^2\theta}{dt^2}.$$

These are expressions for the X and Y components of A's absolute acceleration. From them, we may obtain expressions for the normal and tangential components of A's acceleration, a_n and a_t, along and perpendicular to R, respectively (Fig. 323).

$$a_n = a_x \cos\theta + a_y \sin\theta$$

$$a_n = \cos^2\theta\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right] - 2\frac{dR}{dt}\sin\theta\frac{d\theta}{dt}\cos\theta - R\sin\theta\cos\theta\frac{d^2\theta}{dt^2}$$

$$+ \sin^2\theta\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right] + 2\frac{dR}{dt}\cos\theta\frac{d\theta}{dt}\sin\theta + R\cos\theta\sin\theta\frac{d^2\theta}{dt^2}$$

$$= (\cos^2\theta + \sin^2\theta)\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right]$$

$$= \frac{d^2R}{dt^2} - R\omega^2.$$

$$a_t = a_y \cos\theta - a_x \sin\theta$$

$$a_t = \sin\theta\cos\theta\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right] + 2\frac{dR}{dt}\cos^2\theta\frac{d\theta}{dt} + R\cos^2\theta\frac{d^2\theta}{dt^2}$$

$$- \sin\theta\cos\theta\left[\frac{d^2R}{dt^2} - R\left(\frac{d\theta}{dt}\right)^2\right] + 2\frac{dR}{dt}\sin^2\theta\frac{d\theta}{dt} + R\sin^2\theta\frac{d^2\theta}{dt^2}$$

$$= 2\frac{dR}{dt}\frac{d\theta}{dt}(\cos^2\theta + \sin^2\theta) + R\frac{d^2\theta}{dt^2}(\cos^2\theta + \sin^2\theta)$$

$$= 2\frac{dR}{dt}\frac{d\theta}{dt} + R\frac{d^2\theta}{dt^2}.$$

$\frac{dR}{dt} = v_R$, where v_R is the component of the velocity of point A relative to the coinciding point on the moving radius, in the direction of R.

Then $$a_t = 2\,v_R\omega + R\alpha.$$

If the radius were constant the expression for normal acceleration would reduce to $a_n = \omega^2R$, as in the past, and a_t would equal αR.

The term $2\,v_R\omega$ which has now appeared is known as the *Coriolis'* acceleration. This has resulted from the sliding contact in moving guides, with changing radial distance. Such an acceleration always appears when a particle slides relative to a rotating path.

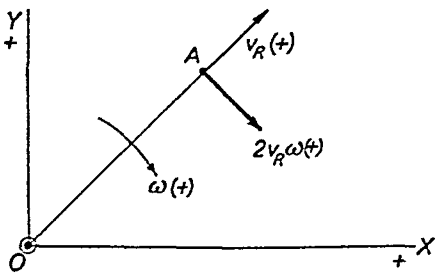


FIG. 324

The sense of the vector $2v_R\omega$ may be determined by comparison with the assumed senses of the basic derivation.

In Fig. 324, the senses assumed

in the basic derivation and the resulting sense of the $2\,v_R\omega$ term are shown.

If either v_R or ω is opposed to the sense shown, the $2\,v_R\omega$ term will be of opposite sense. If both are reversed, the $2\,v_R\omega$ term will again be of the sense shown in Fig. 324.

Illustrative Example. The swinging-block quick-return mechanism of Fig. 325 has a driving crank, BA, which is 1 inch long and has constant angular velocity, $\omega_{BA} = 2$ radians per second, counter-clockwise. $BC = 2$ inches.

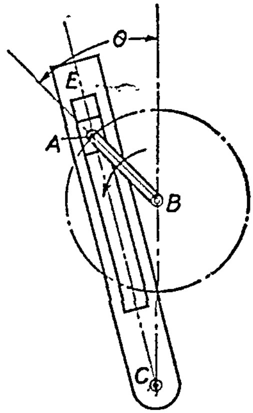


Fig. 325

In the position shown, with $\theta = 45°$, the angular acceleration, α, of the beam CE is to be found.

The velocity of point A (Fig. 326) on crank BA has magnitude $v_1 = \omega_{BA} \times BA = 2 \times 1 = 2$ inches per second, and the direction shown.

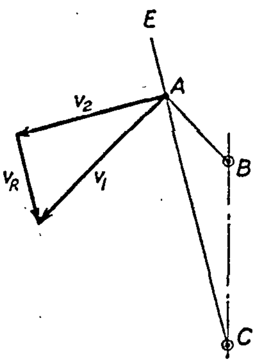


Fig. 326

The velocity of the coinciding point A on CE has magnitude $v_2 = 1.73$ inches per second, and the direction shown.

The velocity of A on BA relative to A on CE has magnitude $v_R = 1.01$ inches per second, and the direction shown.

The angular velocity of CE, $\omega_{CE} = \dfrac{1.73}{2.8} =$ 0.62 radians per second counter-clockwise.

The analysis of accelerations is shown in Fig. 327.

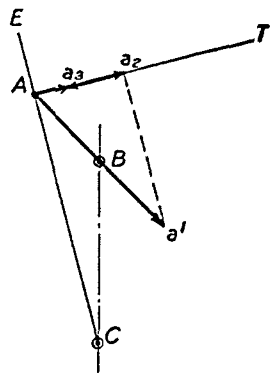


Fig. 327

The T-axis is perpendicular to CE at A.

In this component direction, the basic theorem of absolute and relative motion may be employed: the orthogonal component of the absolute acceleration of point A on BA is equal to the sum of the orthogonal component of the absolute acceleration of point A on CE and the orthogonal component of the acceleration of A on BA relative to A on CE.

A on BA has a resultant absolute acceleration $Aa^1 = (\omega_{BA})^2 \times BA = 2^2 \times 1 = 4$ inches per second per second, in the direction shown.

The orthogonal component on the T-axis of this acceleration is $Aa_2 = 2.02$ inches per second per second, in the direction shown.

Aa_2 equals the vector sum

$$2\, v_R\omega \nrightarrow R\alpha = 2\, v_R\omega_{CE} \nrightarrow (CA \times \alpha_{CE})$$

or,
$$CA \times \alpha_{CE} = Aa_2 \rightarrow 2\, v_R\omega_{CE}$$

$$2\, v_R\omega_{CE} = 2 \times 1.01 \times 0.62 = 1.25 \text{ inches per second per second.}$$

The sense of this vector is from A toward T, and its magnitude is the distance a_3a_2.

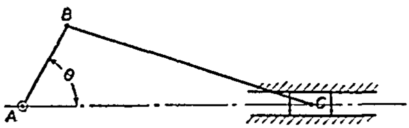
The vector difference is $Aa_3 = R\alpha = CA \times \alpha_{CE} = 0.77$ inches per second per second.

Aa_3 is the tangential component of the acceleration of A on CE, and

$$\alpha_{CE} = \frac{a_t}{R} = \frac{0.77}{2.8} = 0.275 \text{ radians per second per second, clockwise.}$$

PROBLEMS

426. AB has constant angular velocity of 100 r.p.m. clockwise. Determine the absolute acceleration of point C when $\theta = 60°$. $AB = 3.0$ in.; $BC = 8.0$ in.



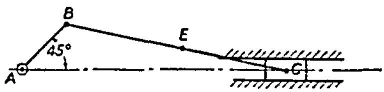
PROB. 426

427. Solve Problem 426 with $\theta = 45°$.

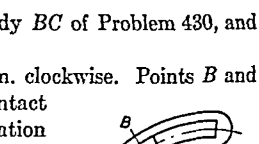
428. Solve Problem 426 with $\theta = 90°$.

429. Solve Problem 426 if AB has angular velocity of 2 radians per sec. clockwise and angular acceleration of 1 radian per sec. per sec. counter-clockwise when $\theta = 60°$.

430. Point C has an absolute velocity of 2 in. per sec., and an absolute acceleration of 2 in. per sec. per sec. both to the right. Determine the angular velocity and angular acceleration of AB. $AB = 2$ in.; $BC = 7$ in.



PROB. 430

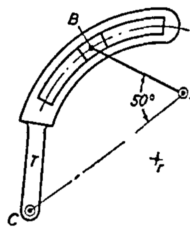


PROB. 432

431. Locate the instantaneous axis of accelerations of body BC of Problem 430, and determine the acceleration of point E, the mid-point on BC.

432. Body AB has constant angular velocity of 200 r.p.m. clockwise. Points B and C are the centers of curvature of the circular surfaces in contact at S. Determine the angular velocity and angular acceleration of body CD, in the position shown. $AB = 6.0$ in.; $AD = 9.0$ in.; $CD = 3.0$ in.; $BS = 1.5$ in. radius; $CS = 2.5$ in. radius.

433. The driving crank AB has an angular velocity of 3 radians per sec. clockwise, and an angular acceleration of 6 radians per sec. per sec. counter-clockwise. Determine the angular velocity and angular acceleration of the large slotted beam T, in the position shown. r is the center of curvature of the slot. $AB = 5.3$ in.; $AC = 9.5$ in.; $Cr = 6.8$ in.; radius of curvature of slot $(Br) = 6.3$ in.



PROB. 433

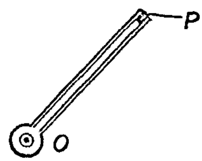
434. Determine the acceleration of point B on AB relative to the coinciding point B on beam T, in Problem 433.

435. If crank AB in Problem 362 rotates with constant angular velocity of 1800 r.p.m., clockwise, determine the absolute accelerations of pistons C and D for the position shown.

436. If crank AB, Problem 363, has a constant angular velocity of 1 radian per min. clockwise, determine the absolute acceleration of F when AB makes an angle of 45° with the horizontal.

437. If piston D, Problem 359, has an absolute velocity of 1 in. per sec. and an absolute acceleration of 2 in. per sec. per sec. both to the right when BC is horizontal, determine the absolute acceleration of F.

438. A particle, P, leaves the tube with a relative velocity (between tube and particle) of 2 f.p.s. The tube rotates about fixed axis O with constant angular velocity of 60 r.p.m. clockwise. Determine the absolute acceleration of the particle at the instant it leaves the tube. $OP = 6$ in



PROB. 438

439. If crank AB, Problem 367, has constant angular velocity of 60 r.p.m. clockwise, determine the absolute accelerations of points D and E when AB makes an angle of 30° with the line marked AS. (The angle is to be measured counter-clockwise from AS about A.)

440. If crank AB, Problem 370, has constant angular velocity of 150 r.p.m., clockwise, determine the absolute accelerations of points E and F, when AB makes an angle of 60° with AS. (The angle is to be measured counter-clockwise from AS about A.)

441. If crank AB, Problem 369, has constant angular velocity of 1 radian per min., clockwise, determine the absolute acceleration of point E, when AB is in its left horizontal position.

111. Acceleratious in Mechanisms. General. The foregoing analyses of accelerations in instantaneous positions of mechanisms supply adequate equipment with which the problems usually encountered may be attacked. When the acceleration property is to be summarized for a series or range of positions, the method of graphical differentiation should be employed. In this case the mechanism is plotted in the series of positions desired, and a displacement-time curve derived. Differentiation of this curve yields the velocity-time relationship and differentiation in turn of the latter curve results in an acceleration-time curve. This method, which has already been discussed, is the only general method which is simple, direct, and applicable readily for series of positions.

TABLE OF CHORDS

Note: Tabulated values give chord for unit radius. In graphical solutions, multiples of unit radius and corresponding chord should be used for greater accuracy.

Deg.	*0′*	*10′*	*20′*	*30′*	*40′*	*50′*	*60′*
0	.0000	.0029	.0058	.0087	.0116	.0145	.0174
1	.0174	.0204	.0233	.0262	.0291	.0320	.0349
2	.0349	.0378	.0407	.0436	.0465	.0494	.0523
3	.0523	.0553	.0582	.0611	.0640	.0669	.0698
4	.0698	.0727	.0756	.0785	.0814	.0843	.0872
5	.0872	.0901	.0930	.0959	.0988	.1017	.1047
6	.1047	.1076	.1105	.1134	.1163	.1192	.1221
7	.1221	.1250	.1279	.1308	.1337	.1366	.1395
8	.1395	.1424	.1453	.1482	.1511	.1540	.1569
9	.1569	.1598	.1627	.1656	.1685	.1714	.1743
10	.1743	.1772	.1801	.1830	.1859	.1888	.1917
11	.1917	.1946	.1975	.2004	.2033	.2062	.2090
12	.2090	.2119	.2148	.2177	.2206	.2235	.2264
13	.2264	.2293	.2322	.2351	.2380	.2409	.2437
14	.2437	.2466	.2495	.2524	.2553	.2582	.2610
15	.2610	.2639	.2668	.2697	.2726	.2755	.2783
16	.2783	.2812	.2841	.2870	.2899	.2927	.2956
17	.2956	.2985	.3014	.3042	.3071	.3100	.3129
18	.3129	.3157	.3186	.3215	.3243	.3272	.3301
19	.3301	.3330	.3358	.3387	.3416	.3444	.3473
20	.3473	.3502	.3530	.3559	.3587	.3616	.3645
21	.3645	.3673	.3702	.3730	.3759	.3788	.3816
22	.3816	.3845	.3873	.3902	.3930	.3959	.3987
23	.3987	.4016	.4044	.4073	.4101	.4130	.4158
24	.4158	.4187	.4215	.4243	.4272	.4300	.4329
25	.4329	.4357	.4385	.4414	.4442	.4471	.4499
26	.4499	.4527	.4556	.4584	.4612	.4641	.4669
27	.4669	.4697	.4725	.4754	.4782	.4810	.4838
28	.4838	.4867	.4895	.4923	.4951	.4979	.5008
29	.5008	.5036	.5064	.5092	.5120	.5148	.5176
30	.5176	.5204	.5232	.5261	.5289	.5317	.5345
31	.5345	.5373	.5401	.5429	.5457	.5485	.5513
32	.5513	.5541	.5569	.5596	.5624	.5652	.5680
33	.5680	.5708	.5736	.5764	.5792	.5820	.5847
34	.5847	.5875	.5903	.5931	.5959	.5986	.6014
35	.6014	.6042	.6069	.6097	.6125	.6153	.6180
36	.6180	.6208	.6236	.6263	.6291	.6318	.6346
37	.6346	.6374	.6401	.6429	.6456	.6484	.6511
38	.6511	.6539	.6566	.6594	.6621	.6649	.6676
39	.6676	.6703	.6731	.6758	.6786	.6813	.6840
40	.6840	.6868	.6895	.6922	.6950	.6977	.7004
41	.7004	.7031	.7059	.7086	.7113	.7140	.7167
42	.7167	.7194	.7222	.7249	.7276	.7303	.7330
43	.7330	.7357	.7384	.7411	.7438	.7465	.7492
44	.7492	.7519	.7546	.7573	.7600	.7627	.7654

TABLE OF CHORDS (*Continued*)

Deg.	0′	10′	20′	30′	40′	50′	60′
45	.7654	.7680	.7707	.7734	.7761	.7788	.7815
46	.7815	.7841	.7868	.7895	.7921	.7948	.7975
47	.7975	.8001	.8028	.8055	.8081	.8108	.8135
48	.8135	.8161	.8188	.8214	.8241	.8267	.8294
49	.8294	.8320	.8347	.8373	.8400	.8426	.8452
50	.8452	.8479	.8505	.8531	.8558	.8584	.8610
51	.8610	.8636	.8663	.8689	.8715	.8741	.8767
52	.8767	.8793	.8820	.8846	.8872	.8898	.8924
53	.8924	.8950	.8976	.9002	.9028	.9054	.9080
54	.9080	.9106	.9132	.9157	.9183	.9209	.9235
55	.9235	.9261	.9286	.9312	.9338	.9364	.9389
56	.9389	.9415	.9441	.9466	.9492	.9518	.9543
57	.9543	.9569	.9594	.9620	.9645	.9671	.9696
58	.9696	.9722	.9747	.9772	.9798	.9823	.9848
59	.9848	.9874	.9899	.9924	.9949	.9975	1.0000
60	1.0000	1.0025	1.0050	1.0075	1.0100	1.0126	1.0151
61	1.0151	1.0176	1.0201	1.0226	1.0251	1.0276	1.0301
62	1.0301	1.0326	1.0350	1.0375	1.0400	1.0425	1.0450
63	1.0450	1.0475	1.0500	1.0524	1.0550	1.0574	1.0598
64	1.0598	1.0623	1.0648	1.0672	1.0697	1.0721	1.0746
65	1.0746	1.0770	1.0795	1.0819	1.0844	1.0868	1.0893
66	1.0893	1.0917	1.0941	1.0966	1.0990	1.1014	1.1039
67	1.1039	1.1063	1.1087	1.1111	1.1135	1.1159	1.1184
68	1.1184	1.1208	1.1232	1.1256	1.1280	1.1304	1.1328
69	1.1328	1.1352	1.1376	1.1400	1.1424	1.1448	1.1471
70	1.1471	1.1495	1.1519	1.1543	1.1567	1.1590	1.1614
71	1.1614	1.1638	1.1661	1.1685	1.1708	1.1732	1.1756
72	1.1756	1.1780	1.1803	1.1862	1.1850	1.1873	1.1896
73	1.1896	1.1920	1.1943	1.1966	1.1990	1.2013	1.2036
74	1.2036	1.2059	1.2083	1.2106	1.2129	1.2152	1.2175
75	1.2175	1.2198	1.2221	1.2244	1.2267	1.2290	1.2313
76	1.2313	1.2336	1.2360	1.2382	1.2405	1.2427	1.2450
77	1.2450	1.2473	1.2496	1.2518	1.2541	1.2564	1.2586
78	1.2586	1.2609	1.2631	1.2654	1.2677	1.2699	1.2721
79	1.2721	1.2744	1.2766	1.2789	1.2811	1.2833	1.2856
80	1.2856	1.2878	1.2900	1.2922	1.2945	1.2967	1.2989
81	1.2989	1.3011	1.3033	1.3055	1.3077	1.3099	1.3121
82	1.3121	1.3143	1.3165	1.3187	1.3209	1.3231	1.3252
83	1.3252	1.3274	1.3296	1.3318	1.3340	1.3361	1.3383
84	1.3383	1.3404	1.3426	1.3447	1.3469	1.3490	1.3510
85	1.3512	1.3533	1.3555	1.3576	1.3597	1.3619	1.3642
86	1.3640	1.3661	1.3682	1.3704	1.3725	1.3746	1.3767
87	1.3767	1.3788	1.3809	1.3830	1.3851	1.3872	1.3893
88	1.3893	1.3914	1.3935	1.3956	1.3977	1.3997	1.4018
89	1.4018	1.4039	1.4060	1.4080	1.4101	1.4121	1.4142
90	1.4142						

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